Spectrum Hierarchies and Subdiagonal Functions

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Abstract

The spectrum of a first-order sentence is the set of cardinalities of its finite models. Relatively little is known about the subclasses of spectra that are obtained by looking only at sentences with a specific signature. In this paper, we study natural subclasses of spectra and their closure properties under simple subdiagonal functions. We show that many natural closure properties turn out to be equivalent to the collapse of potential spectrum hierarchies. We prove all of our results using explicit transformations on first-order structures.

1. Introduction

1.1. Historical Motivation

The *spectrum* of a first-order sentence ϕ is the set of cardinalities of its finite models. Let *Sp*(ϕ) denote the spectrum of ϕ and let *SPEC* denote the set of all spectra.

Let BIN^1 denote the class of spectra of sentences involving a single binary relation symbol. In [5], Fagin asks if every spectrum is in BIN^1. Actually, Fagin restricts the single relation to be irreflexive and symmetric, but we will allow an arbitrary relation in this paper. Fagin also introduces a potential spectrum hierarchy. Let $\mathcal{F}_k$ denote the class of spectra of sentences involving relation symbols of arity at most k. Clearly $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$ for all k. Fagin proposes that this hierarchy is strict, yet no one has been able to prove that it does not collapse to the second level.

Relatively little is known about the classes $\mathcal{F}_k$. Part of the reason that they are not understood is because research on spectra has tended to rely heavily on the techniques of descriptive complexity theory. Many interesting results have been proved by characterizing subclasses of spectra in terms of complexity, and then applying the results of complexity theory [7, 8, 11]. However, no natural characterization of $\mathcal{F}_k$ has been given. As a result, little research has been done on the properties of the classes in Fagin's proposed spectrum hierarchy.

Similarly, very little is known about the classes of spectra obtained by varying the number of binary relation symbols, or the number of unary relation symbols allowed in sentences. In this paper, we will use model-theoretic techniques to explore the properties of these restricted classes of spectra.

1.2. Closure Properties

Given a set of natural numbers S and a function f over the natural numbers, we write $f(S)$ as a shorthand notation for $\{f(n)|n \in S\}$. We say that a class of spectra $\mathcal{S}$ is closed under f if $f(S) \in \mathcal{S}$ whenever $S \in \mathcal{S}$.

Using explicit transformations on graphs, More proves that $\mathcal{F}_2$ is closed under every polynomial with rational coefficients that is assymptotically greater than the identity function [12]. In this paper, we study closure properties under functions that are assymptotically less than the identity function. It turns out that establishing these closure properties is more difficult.

1.3. Summary of Main Results

In order to summarize our results, we introduce a more general notation for subclasses of spectra.

Definition $S \in \mathcal{F}_k^{m_k,\dots,m_1}$ if $S = Sp(\phi)$ for some sentence ϕ which satisfies the following conditions:

1. for each $i \leq k$, ϕ involves at most m_i relation symbols of arity i
2. ϕ involves no other non-logical symbols.

It will be convenient to write $\mathcal{F}_2^p$ as a shorthand for $\mathcal{F}_2^{p,0}$. We remark that $BIN^1 = \mathcal{F}_2^1$.

We study three potential spectrum hierarchies:

(1) $\mathcal{F}_2^{1,0} \subseteq \mathcal{F}_2^{1,1} \subseteq \mathcal{F}_2^{1,2} \subseteq \dots$

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(2) $\mathcal{F}_2^1 \subseteq \mathcal{F}_2^2 \subseteq \mathcal{F}_2^3 \subseteq \dots$

(3) $\mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \mathcal{F}_4 \subseteq \dots$

The first hierarchy is based on the number of unary relation symbols in the sentence and the second hierarchy is based on the number of binary relation symbols. The third hierarchy is Fagin's hierarchy, and it is based on the arity of the relation symbols. The inclusions given above are trivial, but it is not clear if any of them are strict. We remark that (1) is the simplest spectrum hierarchy that is worth studying, because it is well-known that $\mathcal{F}_1$ is the set of finite and co-finite sets.

In this paper, we prove that the collapse of each hierarchy is equivalent to an arithmetic closure condition on the lowest level. In particular, we establish the following results.

- (1) collapses $\iff$ *BIN*^{1} is closed under $n \mapsto n-1$
- (2) collapses $\iff$ *BIN*^{1} is closed under $n \mapsto \lceil \frac{n}{2} \rceil$
- (3) collapses $\iff$ $\mathcal{F}_2$ is closed under $n \mapsto \lceil \sqrt{n} \rceil$.

In the last section of the paper, we turn our attention to the class *BIN*$^1_{bo}$ introduced in [3]. This is the class of spectra of sentences involving a single binary relation symbol with bounded outdegree. More precisely, $S \in$ *BIN*$^1_{bo}$ if $S = Sp(\phi)$ for some ϕ involving a single binary relation symbol R and there is some k such that every model of ϕ assigns R a relation with outdegree bounded by k. Although this class of spectra is clearly a subset of *BIN*^{1}, it is not clear if it is a strict subset. We prove that, if *BIN*^{1} $=$ *BIN*$^1_{bo}$, then Fagin's hierarchy collapses.

Throughout the paper, we will generally assume that all signatures are relational. We remark that some related work has been done on a proposed spectrum hierarchy based on signatures involving only unary function symbols[3, 6].

1.4. Significance of the Results

The most famous problem in the theory of spectra was posed by Asser, when he asked if *SPEC* is closed under complement [1]. This problem has proven to be very difficult, due to a well-known characterization of *SPEC* in terms of complexity theory. Given a set of natural numbers S, there is a corresponding language $\hat{S}$ over the alphabet $\{0, 1\}$ that contains the binary representation of every element of S. It has been shown that $S \in$ *SPEC* if and only if $\hat{S}$ is accepted in exponential time by a non-deterministic Turing machine [4, 9]. Hence, if Asser's problem has a negative solution, it follows by a straightforward argument that $P \neq NP$. This fact has motivated a great deal of research on spectra.

Understanding *BIN*^{1} and the hierarchies we have outlined in this section is important for our understanding of spectra. For example, it has been shown that, if there is a spectrum whose complement is not a spectrum, then there is one in *BIN*^{1} [4]. Establishing general arithmetic closure conditions for *BIN*^{1} and related classes may provide a first step towards an improved understanding of these important subclasses of spectra.

The results that we present give arithmetic conditions for the collapse of some potential spectrum hierarchies. We do not know of any other attempts to systematically study the arithmetic closure properties of *SPEC* under natural subdiagonal functions. We prove our results using simple, definable transformations on first-order structures. This kind of proof was employed in Fagin's early work[5] and more recently this approach was explicitly studied by More[12].

2 Preliminaries

2.1. Definitions and Notation

A finite set of non-logical symbols together with an arity for each symbol will be called a *vocabulary* (or a *signature*). We will assume that all vocabularies contain only relation symbols. Given a vocabulary $\mathcal{L} = \{R_1, \dots, R_m\}$, an $\mathcal{L}$-structure $\mathcal{M} = \langle M, P_1, \dots, P_m \rangle$ is composed of a finite set M called the *universe* and, for each k-ary relation symbol R_i, a relation $P_i \subseteq M^k$ called the *interpretation* of R_i. Note that, by convention, M is used to denote the universe of the structure $\mathcal{M}$. Normally, $R^{\mathcal{M}}$ will denote the interpretation of R in the structure $\mathcal{M}$. Following the conventions of set theory, we identify n with the set $\{0, \dots, n-1\}$ when we are specifying the universe of a structure.

The notation $\bar{a}$ will be used as an abbreviation for a tuple $(a_1, \dots, a_n)$ when the value of n is understood or unimportant. If $\phi(\bar{x})$ is a formula with free variables $\bar{x}$, then we write $\mathcal{M} \models \phi[\bar{a}]$ to indicate that $\bar{a}$ satisfies $\phi(\bar{x})$ in $\mathcal{M}$. The satisfaction relation is defined rigorously in [2].

For some familiar functions f, we use a natural shorthand notation for $f(S)$. For example, if f is $n \mapsto n+1$, then $S+1$ denotes $f(S)$.

2.2. Methodology

Typically, we will start with a vocabulary $\mathcal{L}_1$ and a function f. Given an $\mathcal{L}_1$-spectrum S, we will be interested in proving that $f(S)$ is the spectrum of a sentence in some target vocabulary $\mathcal{L}_2$. For example, we might be interested in proving that, if S is an $\{R\}$-spectrum, then $S+1$ is also an $\{R\}$-spectrum. The prototypical question we are interested in can be stated as follows:

> (*) Given an $\mathcal{L}_1$-spectrum S and a function f whose domain includes S, does it follow that $f(S)$ is an $\mathcal{L}_2$-spectrum?

If $\mathcal{L}_1 = \mathcal{L}_2$, then we are simply asking if the class of $\mathcal{L}_1$-spectra is closed under the function f. Our proofs will be constructive. Starting with an arbitrary $\mathcal{L}_1$-sentence ϕ, we will contruct an $\mathcal{L}_2$-sentence ϕ^* such that $Sp(\phi^*) = f(Sp(\phi))$. We outline the procedure in detail below.

Our methods will be similar to those used in [3] and [12]. Suppose we are given vocabularies $\mathcal{L}_1, \mathcal{L}_2$ and a function f. We want to show that $f(S)$ is an $\mathcal{L}_2$-spectrum whenever S is an $\mathcal{L}_1$-spectrum. Inuitively, we would like to proceed as follows.

(a) construct an injection $\mathcal{M} \mapsto \mathcal{M}^*$ from $\mathcal{L}_1$-structures to $\mathcal{L}_2$-structures such that $|M^*| = f(|M|)$.

(b) construct an injection $\psi \mapsto \psi^*$ from $\mathcal{L}_1$-sentences to $\mathcal{L}_2$-sentences that syntactically captures the transformation on structures

(c) prove that $\mathcal{M} \models \psi$ if and only if $\mathcal{M}^* \models \psi^*$ for every $\mathcal{L}_1$-sentence ψ and every $\mathcal{L}_1$-structure $\mathcal{M}$.

If we could establish (c), then we would have accomplished our goal. Note that (c) requires us to prove a result for all sentences of the vocabulary $\mathcal{L}$. In order to prove a result for all sentences, we typically need to use induction and prove the result more generally for all formulas. Hence, although the procedure outlined above captures the intuitive idea of our proofs, in practice we need to use a slightly more complicated procedure.

Let $\mathcal{L}_1, \mathcal{L}_2$ be vocabularies and let f be a function on the natural numbers. In order to give an affirmative answer to the sample query (*) above, we would procede as follows.

1. let ϕ be an arbitrary $\mathcal{L}_1$-sentence
2. construct an injection $\mathcal{M} \mapsto \mathcal{M}^*$ from $\mathcal{L}_1$-structures to $\mathcal{L}_2$-structures such that $|M^*| = f(|M|)$.
3. construct an injection $(\psi(\bar{x}), \bar{a}) \mapsto (\psi^*_{\bar{a}}(\bar{y}), \bar{a}^*)$ which takes an $\mathcal{L}_1$ formula and a tuple of elements of $\mathcal{M}$ and returns a formula of $\mathcal{L}_2$ and a tuple of elements of $\mathcal{M}^*$.
4. prove that $\mathcal{M} \models \psi[\bar{a}]$ if and only if $\mathcal{M}^* \models \psi^*_{\bar{a}}[\bar{a}^*]$, for every $\mathcal{L}_1$-formula $\psi(\bar{x})$, every $\mathcal{L}_1$-structure $\mathcal{M}$ and every tuple $\bar{a}$ of elements of M.
5. prove that $Sp(\phi^*) = f(Sp(\phi))$.

Although this procedure may seem complex in the abstract, it is a natural technique for proving the results that we want to prove. In practice, we will see that the transformations on structures are intuitive, and the proofs basically fall out of these constructions.

3 Extra Unary relations

We now turn to our proposed spectrum hierarchy based on a single binary relation and an arbitrary number of unary relations. Clearly, for any p, $\mathcal{F}_2^{1,p} \subseteq \mathcal{F}_2^{1,p+1}$. In this section, we prove that determining whether or not this hierarchy collapses to BIN^1 is equivalent to determining if BIN^1 is closed under $n \mapsto n-1$.

Our first theorem states that, if S is an $\mathcal{L}$-spectrum for some $\mathcal{L}$ involving k binary relation symbols, then $S-1$ is an $(\mathcal{L} \cup \{U_1, \ldots, U_{2k}\})$-spectrum, where each U_i is a new unary relation symbol. As an illustration, we prove this theorem in some detail; for subsequent results, we normally provide a simple sketch of the proof.

Theorem 1 *If $S \in \mathcal{F}_2^{k,p}$, then $S-1 \in \mathcal{F}_2^{k,p+2k}$.*

Proof We prove the theorem for $k=1$ and $p=0$; the proof extends easily to the general case. Let $S = Sp(\phi)$ where ϕ involves only one binary relation symbol R.

Given $\mathcal{M} = \langle n, R^{\mathcal{M}} \rangle$, we construct $\mathcal{M}^* = \langle n-1, P^{\mathcal{M}^*}, U_1^{\mathcal{M}^*}, U_2^{\mathcal{M}^*} \rangle$, where $P^{\mathcal{M}^*}$ is binary and $U_1^{\mathcal{M}^*}, U_2^{\mathcal{M}^*}$ are unary. For all $a, b < n-1$, let

$$\begin{aligned} P^{\mathcal{M}^*}ab &\iff R^{\mathcal{M}}ab \\ U_1^{\mathcal{M}^*}a &\iff R^{\mathcal{M}}a(n-1) \\ U_2^{\mathcal{M}^*}a &\iff R^{\mathcal{M}}(n-1)a. \end{aligned}$$

Observe that, given $\mathcal{M}^*$ and $(a,b) \in n \times n$ such that either $a \neq n-1$ or $b \neq n-1$, it is possible to determine whether or not $(a,b) \in R^{\mathcal{M}}$. However, there is no way of determining from $\mathcal{M}^*$ whether or not $(n-1, n-1) \in R^{\mathcal{M}}$.

Given a t-tuple $\bar{a} = (a_1, \ldots, a_t) \in n^t$, let $v(\bar{a}) = (v(a_1), \ldots, v(a_t))$ denote the unique t-tuple in $\{0,1\}^t$ such that $v(a_i) = 1$ if and only if $a_i = n-1$. Let $\psi(\bar{x})$ be an $\{R\}$-formula. Relative to $v(\bar{a})$, there is a natural translation $\psi^0_{v(\bar{a})}(\bar{x})$ of $\psi(\bar{x})$ into the vocabulary $\{P, U_1, U_2\}$. Intuitively, we want $\psi^0_{v(\bar{a})}(\bar{x})$ to be a formula that is satisfied by $\bar{a}$ in $\mathcal{M}^*$ if and only if $\psi(\bar{x})$ is satisfied by $\bar{a}$ in $\mathcal{M}$. For example, if $v(a_i) = 1$ and $v(a_j) < 0$, then one step in the translation would be to replace all occurrences of Rx_ix_j with the formula U_2x_j, because $\mathcal{M}^*$ is constructed so that $U_2^{\mathcal{M}^*}a_j$ if and only if $R^{\mathcal{M}}(n-1)(a_j)$. The only problem is that there is no appropriate replacement for Rx_ix_j when $v(a_i) = v(a_j) = 1$. However, if we assume that $R^{\mathcal{M}}(n-1)(n-1)$, then all occurrences of Rx_ix_j can simply be replaced by $x_i = x_i$ in this case.

Let $\bar{u} = (u_1, \ldots, u_t) \in \{0,1\}^t$. Define the formula $\psi^0_{\bar{u}}(\bar{x})$ recursively as follows:

1. If $\psi(\bar{x})$ is $x_i = x_j$ for some i, j, $1 \leq i, j \leq t$:
   - If $u_i = u_j = 0$, then $\psi^0_{\bar{u}}(\bar{x})$ is $x_i = x_j$.
   - If $u_i = u_j = 1$, then $\psi^0_{\bar{u}}(\bar{x})$ is $x_i = x_i$.

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- If $u_i \neq u_j$, then $\psi^0_{\bar{u}}(\bar{x})$ is $x_i \neq x_i$.

2. If $\psi(\bar{x})$ is Rx_ix_j for some $i, j,\ 1 \leq i, j \leq t$:

   - If $u_i = 0$ and $u_j = 0$, then $\psi^0_{\bar{u}}(\bar{x})$ is Px_ix_j.
   - If $u_i = 0$ and $u_j = 1$, then $\psi^0_{\bar{u}}(\bar{x})$ is U_1x_i.
   - If $u_i = 1$ and $u_j = 0$, then $\psi^0_{\bar{u}}(\bar{x})$ is U_2x_j.
   - If $u_i = 1$ and $u_j = 1$, then $\psi^0_{\bar{u}}(\bar{x})$ is $x_i = x_i$.

3. If $\psi(\bar{x})$ is $\neg\theta(\bar{x})$:

   - $\psi^0_{\bar{u}}(\bar{x})$ is $\neg\theta^0_{\bar{u}}(\bar{x})$.

4. If $\psi(\bar{x})$ is $\theta(\bar{x}) \wedge \gamma(\bar{x})$:

   - $\psi^0_{\bar{u}}(\bar{x})$ is $\theta^0_{\bar{u}}(\bar{x}) \wedge \gamma^0_{\bar{u}}(\bar{x})$.

5. If $\psi(\bar{x})$ is $(\exists x_{t+1}\theta)(\bar{x})$:

   - $\psi^0_{\bar{u}}(\bar{x})$ is $(\exists x_{t+1}(\theta^0_{\bar{u}\wedge 0} \vee \theta^0_{\bar{u}\wedge 1}))(\bar{x})$.

By induction on $\psi(\bar{x})$, we prove the following proposition:

Proposition 1 *Let $\mathcal{M} = \langle n, R^{\mathcal{M}}\rangle$ where $n > 1$ and $R^{\mathcal{M}}(n-1)(n-1)$. For every $\{R\}$-formula $\psi(\bar{x})$, every $\bar{a} \in n^t$ and every $\bar{b} \in (n-1)^t$ such that $a_i = b_i$ whenever $a_i < n-1$, we have*

$$\mathcal{M} \models \psi[\bar{a}] \iff \mathcal{M}^* \models \psi^0_{v(\bar{a})}[\bar{b}].$$

In each stage of the induction, assume that $\psi(\bar{x})$ is an $\{R\}$-formula, $\bar{a} \in n^t$, $\bar{b} \in (n-1)^t$ and $a_i = b_i$ whenever $a_i < n-1$.

Suppose $\psi(\bar{x})$ is $x_i = x_j$. If $a_i, a_j < n-1$, then clearly $\mathcal{M} \models a_i = a_j$ if and only if $\mathcal{M}^* \models b_i = b_j$, since $b_i = a_i$ and $b_j = a_j$. If $a_i = a_j = n-1$, then $\mathcal{M} \models a_i = a_j$ and $\mathcal{M}^* \models b_i = b_i$. If $a_i = n-1$ and $a_j < n-1$ (or $a_i < n-1$ and $a_j = n-1$), then $\mathcal{M} \not\models a_i = a_j$ and $\mathcal{M}^* \not\models \psi^0_{v(\bar{a})}[a_i, a_j]$, since $\psi^0_{v(\bar{a})}(x_i, x_j)$ is $x_i \neq x_i$. So the result holds for equality formulas.

Suppose $\psi(\bar{x})$ is Rx_ix_j. Then

$$\begin{aligned} R^{\mathcal{M}}a_ia_j \quad &\iff \quad a_i, a_j < n-1 \text{ and } P^{\mathcal{M}^*}a_ia_j \\ &\qquad\ \text{or } a_i < a_j = n-1 \text{ and } U_1^{\mathcal{M}^*}a_i \\ &\qquad\ \text{or } a_j < a_i = n-1 \text{ and } U_2^{\mathcal{M}^*}a_j \\ &\qquad\ \text{or } a_i = a_j = n-1 \\ &\iff \quad \mathcal{M}^* \models \psi^0_{v(\bar{a})}[\bar{b}]. \end{aligned}$$

It is clear that the result holds for negations and conjunctions. Now suppose that $\psi(\bar{x})$ is $(\exists x_{t+1}\theta)(\bar{x})$ and assume that Proposition 1 holds for $\theta(\bar{x}, x_{t+1})$.

$$\begin{aligned} & \mathcal{M} \models (\exists x_{t+1}\theta)[\bar{a}] \\ \iff\ & \text{for some } c < n, \mathcal{M} \models \theta[\bar{a}, c] \\ \iff\ & \text{for some } c < n-1, \mathcal{M} \models \theta[\bar{a}, c] \\ & \text{or } \mathcal{M} \models \theta[\bar{a}, n-1] \\ \iff\ & \text{for some } c < n-1, \mathcal{M}^* \models \theta^0_{v(\bar{a})\wedge 0}[\bar{b}, c] \\ & \text{or } \mathcal{M}^* \models \theta^0_{v(\bar{a})\wedge 1}[\bar{b}, c] \\ \iff\ & \mathcal{M}^* \models (\exists x_{t+1}(\theta^0_{v(\bar{a})\wedge 0} \vee \theta^0_{v(\bar{a})\wedge 1}))[\bar{b}]. \end{aligned}$$

This completes the proof of Proposition 1.

Next, we define a mapping that intuitively represents a translation of an $\{R\}$-formula into the vocabulary $\{P, U_1, U_2\}$ provided that $(n-1, n-1) \notin R^{\mathcal{M}}$. Given an $\{R\}$-formula $\psi(\bar{x})$ and a t-tuple $\bar{u} \in \{0,1\}^t$, let $\psi^1_{\bar{u}}(\bar{x})$ be defined recursively in exactly the same way as $\psi^0_{\bar{u}}(\bar{x})$ except that the last clause of step 2 is replaced by

- If $u_i = 1$ and $u_j = 1$, then $\psi^1_{\bar{u}}(x_1, \ldots, x_t)$ is $x_i \neq x_i$.

By an induction similar to the proof of Proposition 1, one can prove the next result.

Proposition 2 *Let $\mathcal{M} = \langle n, R^{\mathcal{M}}\rangle$ where $n > 1$ and not $R^{\mathcal{M}}(n-1)(n-1)$. For every $\{R\}$-formula $\psi(\bar{x})$, every $\bar{a} \in n^t$ and every $\bar{b} \in (n-1)^t$ such that $a_i = b_i$ whenever $a_i < n-1$, we have*

$$\mathcal{M} \models \psi[\bar{a}] \iff \mathcal{M}^* \models \psi^1_{v(\bar{a})}[\bar{b}].$$

We remark that the constructions outlined above define mappings on sentences that do not rely on any particular tuple of elements. We now prove that $Sp(\phi^0 \vee \phi^1) = S - 1$.

Suppose that $n \in S-1$, so there exists $\mathcal{M} = \langle n+1, R^{\mathcal{M}}\rangle$ such that $\mathcal{M} \models \phi$. By Proposition 1, if $R^{\mathcal{M}}nn$ then $\mathcal{M}^* \models \phi^0$. Otherwise, by Proposition 2, $\mathcal{M}^* \models \phi^1$. Hence $\mathcal{M}^* \models \phi^0 \vee \phi^1$, so $n \in Sp(\phi^0 \vee \phi^1)$.

Suppose that $n \in Sp(\phi^0 \vee \phi^1)$ and say $\mathcal{A} = \langle n, P^{\mathcal{A}}, U_1^{\mathcal{A}}, U_2^{\mathcal{A}}\rangle \models \phi^0 \vee \phi^1$. If $\mathcal{A} \models \phi^0$, then define $\mathcal{M} = \langle n+1, R^{\mathcal{M}}\rangle$ as follows:

$$\begin{aligned} R^{\mathcal{M}}ab \quad \iff \quad & a, b < n \text{ and } P^{\mathcal{A}}ab \\ & \text{or } a < n, b = n \text{ and } U_1^{\mathcal{A}}a \\ & \text{or } a = n, b < n \text{ and } U_2^{\mathcal{A}}a \\ & \text{or } a = b = n. \end{aligned}$$

Hence, $\mathcal{A} = \mathcal{M}^*$ and $R^{\mathcal{M}}nn$. By Proposition 1, $\mathcal{M} \models \phi$. Similarly, if $\mathcal{A} \models \phi^1$ then we can define $\mathcal{M}$ so that $\mathcal{M} \models \phi$ and $M = n+1$. Therefore, $n+1 \in Sp(\phi)$ which means that $n \in S-1$. This completes the proof. □

We prove the converse of Theorem 1.

Theorem 2 *If $S \in \mathcal{F}_2^{k,p+2k}$, then $S+1 \in \mathcal{F}_2^{k,p}$.*

Proof We sketch the case $k = 1, p = 0$.

In this case, we simply want to reverse the tranformation on structures from the previous proof. We start with an $\{R, U_1, U_2\}$-structure, and we would like to encode it in an $\{R\}$-structure with one additional element. This is straightforward: the interpretation of R on the addional element can be used to pick out the interpretations of U_1 and U_2 in the original model.

The simple transformation on structures in this case leads directly to a simple translation on formulas. As an intermediate step we introduce a constant symbol for the additional element c. We translate $\{R, U_1, U_2\}$-formulas by relativizing all quantifiers to the set of elements other than c, we replace U_1x with Rxc, and we replace U_2x with Rcx. Given this translation, we can prove our desired result by a simple induction. □

By combining Theorems 1 and 2, we get a nice biconditional relationship:

$$S \in \mathcal{F}_2^{k,p+2k} \iff S + 1 \in \mathcal{F}_2^{k,p}. \qquad (1)$$

This result is particularly satisfying because the classes involved are both natural spectrum classes that are constructed by simply placing restrictions on the vocabulary of the sentences that define them. In the following sections, we need to have a slightly more complicated condition defining some of the spectrum classes which we discuss.

As a simple consequence of (1), we get a corresponding result for the function $n \mapsto n + l$ for any l.

Corollary 1 *$S \in \mathcal{F}_2^{k,p+2kl}$ if and only if $S + l \in \mathcal{F}_2^{k,p}$.*

Proof Immediate. □

We are now in a position to prove a theorem that relates the closure of *BIN*^{1} under $n \mapsto n - 1$ with the collapse of the spectrum hierarchy based on the number of extra unary relation symbols.

Theorem 3 *The following are equivalent:*

1. *BIN*^{1} *is closed under* $n \mapsto n - 1$
2. *BIN*$^1 = \mathcal{F}_2^{1,2}$
3. *BIN*$^1 = \bigcup_{i=0}^{\infty} \mathcal{F}_2^{1,i}$

Proof (1 ⇒ 3) Assume *BIN*^{1} is closed under $n \mapsto n - 1$. Let $l \in \omega$ and let ϕ involve one binary relation symbol and $2l$ unary relation symbols. By Corollary 1, $Sp(\phi) + l \in$ *BIN*^{1}. Now apply the hypothesis l times to the set $Sp(\phi) + l$ to get $Sp(\phi) \in$ *BIN*^{1}.

(3 ⇒ 2) Immediate.

(2 ⇒ 1) Assume the hypothesis and suppose $S \in$ *BIN*^{1}. By Theorem 1, $S - 1 \in \mathcal{F}_2^{1,2}$. Therefore $S - 1 \in$ *BIN*^{1}. □

4 Number of Binary Relation Symbols

In this section we are concerned with the hypothetical spectrum hierarchy generated by looking at classes of spectra of sentences involving k binary relation symbols, for each $k \in \omega$. The following definition extends the notation introduced in the introduction.

Definition Let σ be a first-order sentence. $S \in \mathcal{F}_k^{m_k,\dots,m_1}[\sigma]$ if $S = Sp(\sigma \wedge \phi)$ for some sentence ϕ which satisfies the following conditions:

1. ϕ may involve any of the non-logical symbols in σ
2. for each $i \le k$, ϕ involves at most m_i relation symbols of arity i other than the relation symbols in σ
3. ϕ involves no other non-logical symbols.

We think of these as relativized spectrum classes.

For the rest of this section, let Ω denote the following sentence:

$$\begin{aligned}\forall x \exists y \quad & [(Qxy \vee Qyx) \\ & \wedge \neg(Qxy \wedge Qyx) \\ & \wedge \forall z \neq y(\neg Qxz \wedge \neg Qzx)].\end{aligned}$$

Notice that Ω is true in a structure $\mathcal{M}$ just in case the relation $Q^{\mathcal{M}}$ divides the universe into $|M|/2$ ordered pairs such that every element of M is in exactly one pair.

There are two principal results in this section. The first result is the following:

$$S \in \mathcal{F}_2^{k,0}[\Omega] \iff S/2 \in \mathcal{F}_2^{4k,0}. \qquad (2)$$

We obtain this result by mapping models of Ω with size $2n$ to structures with size n where each element of the smaller structure intuitively represents an ordered pair of elements from the larger structure.

The second principal result gives a closure property that is equivalent to the collapse to *BIN*^{1} of the hierarchy on $\mathcal{F}_2$ based on the number of relation symbols. For any rational number q, let $\lceil q \rceil$ denote the least integer greater than or equal to q. We prove that *BIN*^{1} is closed under $n \mapsto \lceil \frac{n}{2} \rceil$ if and only if *BIN*$^1 = \mathcal{F}_2$.

Theorem 4 *If $S \in \mathcal{F}_2^{k,0}[\Omega]$, then $S/2 \in \mathcal{F}_2^{4k,0}$.*

Proof We remark that, if S satisfies the premise of the theorem, then every element of S is even. Hence $S/2$ will be a well-defined set of natural numbers.

We consider the case when $k = 1$, because the proof is essentially the same for higher values of k. Let $S = Sp(\Omega \wedge \phi)$ for some $\{Q, R\}$-sentence ϕ. Given

$\mathcal{M} = \langle 2m, Q^{\mathcal{M}}, R^{\mathcal{M}} \rangle$, define the structure $\mathcal{M}^* = \langle Q^{\mathcal{M}}, F_1^{\mathcal{M}^*}, F_2^{\mathcal{M}^*}, F_3^{\mathcal{M}^*}, F_4^{\mathcal{M}^*} \rangle$ by:

$$\begin{array}{lcl} F_1^{\mathcal{M}^*}(a,b)(c,d) & \iff & R^{\mathcal{M}}ac \\ F_2^{\mathcal{M}^*}(a,b)(c,d) & \iff & R^{\mathcal{M}}bc \\ F_3^{\mathcal{M}^*}(a,b)(c,d) & \iff & R^{\mathcal{M}}ad \\ F_4^{\mathcal{M}^*}(a,b)(c,d) & \iff & R^{\mathcal{M}}bd. \end{array}$$

Note that the universe of $\mathcal{M}^*$ is the subset of $2m \times 2m$ which is the interpretation of the relation symbol Q in $\mathcal{M}$.

In order to complete the proof, we need to create a translation from $\{Q, R\}$-formulas to $\{F_1, F_2, F_3, F_4\}$-formulas. Intuitively, it is clear that this is possible. The idea is simply to use the F_i relation symbols to give a component-wise translation of R. We omit the details of the translation, because they are somewhat long for the present work. □

We now state a useful corollary. In the proof, we use the standard notation $\phi^{\psi(x)}$ to denote the relativization of ϕ under $\psi(x)$.

Corollary 2 *$\mathcal{F}_2$ is closed under $n \mapsto \lceil \frac{n}{2} \rceil$.*

Proof Let $S = Sp(\phi) \in \mathcal{F}_2$. Notice that $Sp(\Omega) = \{2n : n \in \omega\}$. Let U be a new unary relation symbol and consider the sentence $\Omega \wedge (\phi \vee (\exists! yUy \wedge \phi^{\neg Ux}))$. The spectrum of this sentence is

$$T = \{2n : n \in \omega\} \cap \{n : n \in S \text{ or } n-1 \in S\}$$

Observe that $T \in \mathcal{F}_2[\Omega]$, so by Theorem 4, $T/2 \in \mathcal{F}_2$. But $\lceil S/2 \rceil = T/2$, so $\lceil S/2 \rceil \in \mathcal{F}_2$. □

We prove the converse of Theorem 4.

Theorem 5 *If $S \in \mathcal{F}_2^{4k,0}$, then $2S \in \mathcal{F}_2^{k,0}[\Omega]$.*

Proof Consider the case where $k = 1$. Given $\mathcal{M} = \langle n, F_1^{\mathcal{M}}, F_2^{\mathcal{M}}, F_3^{\mathcal{M}}, F_4^{\mathcal{M}} \rangle$, we construct a new structure $\mathcal{M}^* = \langle 2n, Q^{\mathcal{M}^*}, R^{\mathcal{M}^*} \rangle$. The binary relation $Q^{\mathcal{M}^*}$ is $\{(a, a+n) : a < n\}$, and the binary relation $R^{\mathcal{M}^*}$ is defined as follows:

$$\begin{array}{lcl} R^{\mathcal{M}^*}ab & \iff & a < n, b < n, \text{ and } F_1^{\mathcal{M}}ab \\ & & n \leq a < 2n, b < n, \text{ and } F_2^{\mathcal{M}}(a-n)b \\ & & a < n, n \leq b < 2n, \text{ and } F_3^{\mathcal{M}}a(b-n) \\ & & n \leq a, b < 2n, \text{ and } F_4^{\mathcal{M}}(a-n)(b-n) \end{array}$$

The translation on formulas is a straightforward syntactic representation of this transformation on structures. □

By combining the last two theorems, we get the following relationship:

$$S \in \mathcal{F}_2^{4k,0} \iff 2S \in \mathcal{F}_2^{k,0}[\Omega].$$

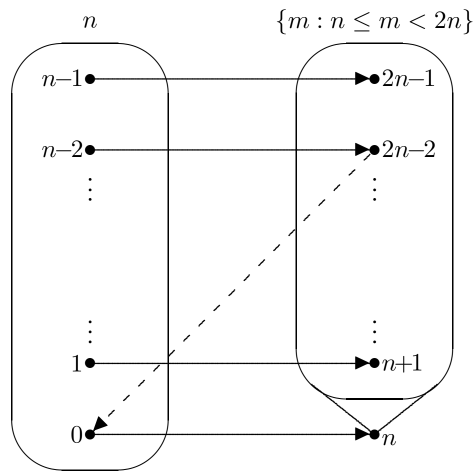

Figure 1.

We remark that, using our methods, it does not seem to be possible to prove this result if the class $\mathcal{F}_2^{k,0}[\Omega]$ is replaced by $\mathcal{F}_2^{k,0}$.

The following corollary is immediate.

Corollary 3 *If $S \in \mathcal{F}_2^{4k,0}$, then $2S \in \mathcal{F}_2^{k+1,0}$.*

We would like to use this corollary to relate the closure of BIN^1 under $n \mapsto \lceil \frac{n}{2} \rceil$ with the collapse of the spectrum hierarchy based on the number of binary relations. We need one more tool.

Lemma 1 *If $S \in \mathcal{F}_2^{2,0}$, then $2S \in BIN^1$.*

Proof Let $S = Sp(\phi)$ for some sentence ϕ involving two binary relation symbols Q and R. Let $\mathcal{M} = \langle n, Q^{\mathcal{M}}, R^{\mathcal{M}} \rangle$. The idea is to define $\mathcal{M}^* = \langle 2n, P^{\mathcal{M}^*} \rangle$ so that it divides the universe into two disjoint components and it acts like $R^{\mathcal{M}}$ on one and $Q^{\mathcal{M}}$ on the other.

The basic picture of $\mathcal{M}^*$ is shown in Figure 1. The arrows in Figure 1 represent how the interpretation of $P^{\mathcal{M}^*}$ is used to partition the universe into two sets. On the left side $P^{\mathcal{M}^*}$ behaves like $Q^{\mathcal{M}}$ and on the right side $P^{\mathcal{M}^*}$ behaves like $R^{\mathcal{M}}$. We use the element n to partition the universe. Specifically, if $(n, a) \in P^{\mathcal{M}^*}$, then a is on the $R^{\mathcal{M}}$-side. Otherwise a is on the $Q^{\mathcal{M}}$-side.

The main problem that must be addressed is related to the element n. Since n must be used to define a particular subset of the universe, the edges from n cannot encode the relation $R^{\mathcal{M}}$ on the element n. However, as indicated in Figure 1, the element 0 is the only element from the left side that is related to n. This means that 0 is syntactically definable, and it can be used to encode the behaviour of $R^{\mathcal{M}}$ at n. In the interest of space, we omit the details of the proof. □

A more general version of Lemma 1 would say that $kS \in \mathit{BIN}^1$ whenever $S \in \mathcal{F}_2^{k,0}$. We have not proved this general version because the combination of the existing lemma with Theorem 5 gives a stronger result in the sense that, for large k, there will always be a $p < k$ such that $pS \in \mathit{BIN}^1$. We also remark that the next theorem could be proved without Theorem 5 by appealing to a more general version of Lemma 1. However, Theorem 5 was needed to give a simple characterization of $\mathcal{F}_2^{4,0}$ which would not have followed from a generalized lemma.

Theorem 6 *BIN^1 is closed under $n \mapsto \lceil \frac{n}{2} \rceil$ if and only if $\mathit{BIN}^1 = \mathcal{F}_2$.*

Proof ($\Leftarrow$) Assume $\mathit{BIN}^1 = \mathcal{F}_2$, and suppose $S \in \mathit{BIN}^1$. By Corollary 2, $\lceil \frac{S}{2} \rceil \in \mathcal{F}_2 = \mathit{BIN}^1$.

($\Rightarrow$) Assume BIN^1 is closed under $n \mapsto \lceil \frac{n}{2} \rceil$, and suppose $S \in \mathcal{F}_2$. By repeatedly applying Corollary 3, there is some p such that $2^p S \in \mathcal{F}_2^{2,0}$. Then by Lemma 1, $2^{p+1} S \in \mathit{BIN}^1$. By the hypothesis it follows that $S \in \mathit{BIN}^1$. □

5 Arity of Relation Symbols

In [5], Fagin suggests that, for every k, $\mathcal{F}_k \subsetneq \mathcal{F}_{k+1}$ but he is unable to prove it for any $k > 1$. In this section, we consider some arithmetic relationships between different levels of Fagin's proposed spectrum hierarchy.

To prove the first result of this section, we need to define a sentence Λ in the vocabulary of two binary relation symbols F_1 and F_2. We define Λ in such a way that $\mathcal{M} \models \Lambda$ if and only if the following conditions hold:

1. for each $(a, b) \in \{a : F_1 aa\}^2$, there is a unique point $m \in M$ such that $F_1 am$ and $F_2 bm$
2. for each $m \in M$, there is a unique pair $(a, b) \in \{a : F_1 aa\}^2$ such that $F_1 am$ and $F_2 bm$.

Let Λ denote the following sentence:

$$\begin{aligned} &\forall w \exists! x \exists! y (F_1 xx \wedge F_1 yy \wedge F_1 xw \wedge F_2 yw \\ &\qquad \wedge \forall z ((F_1 xz \wedge F_2 yz) \rightarrow w = z)) \\ &\wedge \forall x \forall y\, ((F_1 xx \wedge F_1 yy) \rightarrow \exists! z (F_1 xz \wedge F_2 yz \\ &\qquad \wedge \forall u \forall v ((F_1 uz \wedge F_2 vz) \rightarrow (u = x \wedge v = y)). \end{aligned}$$

Observe that every model of Λ will have cardinality m^2 where m is the number of points that satisfy $F_1 xx$. Intuitively, whenever $\mathcal{M} \models \Lambda$, we think of M as a set of pairs

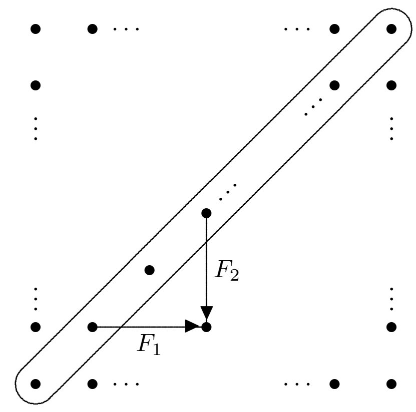

Figure 2.

where $\{a : F_1 aa\}$ is the set of diagonal elements of M and $F_i ab$ if and only if a is a diagonal element and the i^{th} coordinate of a is the i^{th} coordinate of b.

Figure 2 provides an illustration of a model of Λ. The points enclosed in the curve represent the set $\{a : F_1 aa\}$.

One of the principal results in this section is a proof that

$$S \in \mathcal{F}_4^{k,0,0,0} \iff S^2 \in \mathcal{F}_2^{k,0}[\Lambda].$$

To prove this result, we use the fact that every model of Λ is isomorphic to a structure with universe $n \times n$ in which the usual coordinatization is definable.

There is a natural bijection between binary structures with universe $n \times n$ and quaternary structures with universe n. By formalizing this intuitive observation, we can prove our desired result.

Theorem 7 *If $S \in \mathcal{F}_4^{k,0,0,0}$, then $S^2 \in \mathcal{F}_2^{k,0}[\Lambda]$.*

Proof We prove the case $k = 1$. Let $S = \mathit{Sp}(\phi)$, where ϕ involves one quaternary relation symbol R. Given $\mathcal{M} = \langle n, R^{\mathcal{M}} \rangle$, define $\mathcal{M}^* = \langle n \times n, F_1^{\mathcal{M}^*}, F_2^{\mathcal{M}^*}, P^{\mathcal{M}^*} \rangle$ as follows:

$$\begin{aligned} F_1^{\mathcal{M}^*}(a,b)(c,d) &\iff a = b = c \\ F_2^{\mathcal{M}^*}(a,b)(c,d) &\iff a = b = d \\ P^{\mathcal{M}^*}(a,b)(c,d) &\iff R^{\mathcal{M}} abcd. \end{aligned}$$

Note that $\{a : F_1^{\mathcal{M}^*} aa\} = \{(a,a) : a < n\}$.

Let $\psi(\bar{x})$ be a formula in the vocabulary $\{R\}$. We define a formula $\psi^*(\bar{x})$ such that, for every $\bar{a} = (a_1, \ldots, a_t) \in n^t$,

$$\mathcal{M} \models \psi[\bar{a}] \iff \mathcal{M}^* \models \psi^*[(a_1, a_1), \ldots, (a_t, a_t)]. \quad (3)$$

The formula $\psi^*(\bar{x})$ is the conjunction of Λ with the formula $\psi_1(\bar{x})$ that we define presently. Let $\psi_1(\bar{x})$ be obtained from $\psi(\bar{x})$ by relativizing all the quantifiers to the set $\{a : F_1aa\}$ and by replacing all occurrences of $Rwxyz$ with the following:

$$\exists u \exists v \quad (F_1ww \wedge F_1xx \wedge F_1yy \wedge F_1zz \\ \wedge F_1wu \wedge F_2xu \wedge F_1yv \wedge F_2zv \wedge Puv).$$

It is now possible to prove (3) by a straightforward induction on $\psi(\bar{x})$.

The last step is to show that $Sp(\phi^*) = S^2$. Suppose $n \in S^2$. Since $\sqrt{n} \in S$, it follows immediately from (3) that $n \in Sp(\phi^*)$.

Suppose that $n \in Sp(\phi^*)$. Since Λ is a conjunct of ϕ, there exists $\mathcal{A} = \langle \sqrt{n} \times \sqrt{n}, F_1^{\mathcal{A}}, F_2^{\mathcal{A}}, P^{\mathcal{A}} \rangle$ such that $\mathcal{A} \models \phi^*$ and, the following conditions hold:

1. $\{(a,b) : F_1^{\mathcal{A}}(a,b)\} = \{(a,a) : a < n\}$
2. $F_1^{\mathcal{A}}(a,b)(c,d) \iff a = b = c$
3. $F_2^{\mathcal{A}}(a,b)(c,d) \iff a = b = d$

Define the structure $\mathcal{M} = \langle \sqrt{n}, R^{\mathcal{M}} \rangle$ as follows:

$$R^{\mathcal{M}}abcd \iff P^{\mathcal{A}}(a,b)(c,d).$$

Clearly $\mathcal{A} = \mathcal{M}^*$, so it follows from (3) that $\mathcal{M} \models \phi$. Therefore $\sqrt{n}^2 = n \in S^2$. □

Fagin proves a result similar to Theorem 7 in [5]. In particular, Fagin uses model-theoretic techniques to prove that $S^k \in BIN^1$ whenever $S \in \mathcal{F}_k$. Under the premise of Theorem 7, Fagin's result allows us to conclude that $S^4 \in BIN^1$. Hence, although our result requires more relation symbols in the target vocabulary, it uses a smaller power to bring the spectrum into $\mathcal{F}_2$.

We now prove the converse of Theorem 7.

Theorem 8 *If $S \in \mathcal{F}_2^{k,0}[\Lambda]$, then $\sqrt{S} \in \mathcal{F}_4^{k,0,0,0}$.*

Proof Once again, we prove only the case $k = 1$.

Suppose that $S = Sp(\Lambda \wedge \phi)$ where ϕ involves F_1, F_2, and P. Note that, whenever $\mathcal{A} \models \Lambda$, it must be the case that $|A| = n^2$ for some n. Given $\mathcal{M} = \langle n \times n, F_1^{\mathcal{M}}, F_2^{\mathcal{M}}, P^{\mathcal{M}} \rangle$, define $\mathcal{M}^* = \langle n, R^{\mathcal{M}^*} \rangle$ as follows:

$$R^{\mathcal{M}^*}abcd \iff P^{\mathcal{M}}(a,b)(c,d)$$

Let ψ be an arbitrary $\{F_1, F_2, P\}$-formula. The formula ψ^* is obtained by making the following changes to ψ:

1. replace $\exists x$ with $\exists x_1 \exists x_2$
2. replace $x = y$ with $x_1 = y_1 \wedge x_2 = y_2$
3. replace F_1xy with $x_1 = x_2 \wedge x_1 = y_1$
4. replace F_2xy with $x_1 = x_2 \wedge x_1 = y_2$
5. replace Pxy with $Rx_1x_2y_1y_2$.

The details of the proof fall directly out of this construction. □

We remark that the basic idea for the proof of Theorem 8 is used by Fagin to prove that $\{n : n^k \in S\} \in SPEC$ whenever $S \in SPEC$ [5]. By combining Theorems 7 and 8, we get the following relationship:

$$S \in \mathcal{F}_4^{k,0,0,0} \iff S^2 \in \mathcal{F}_2^{k,0}[\Lambda].$$

We now formulate a more general version of this relationship.

Consider the sentence Λ. Intuitively, Λ asserts that there is a definable 1-1 correspondence between the universe and the set $\{a : F_1aa\}^2$. Let $p \in \omega$ and, for each $i \leq p$, let F_i^p denote a p-ary relation symbol. It is clear that there is a sentence Λ_p which asserts that there is a definable 1-1 correspondence between the universe and the set $\{a : F_1^p a \dots a\}^p$. The following theorem is a generalization of Theorems 7 and 8. For simplicity, we let $\bar{0}$ denote a tuple of 0's of indeterminate length.

Theorem 9 $S \in \mathcal{F}_{p^2}^{k,\bar{0}}$ *if and only if* $S^p \in \mathcal{F}_p^{k,\bar{0}}[\Lambda_p]$

Proof The proof of the 'if' part is an easy generalization of the proof of Theorem 7 and the proof of the 'only if' part is an easy generalization of the proof of Theorem 8. □

Corollary 4 *If $S \in \mathcal{F}_{p^2}$ then $S^p \in \mathcal{F}_p$.*

Proof Immediate. □

We conclude this section with an observation that relates the collapse of Fagin's arity hierarchy to an arithmetic closure property.

Theorem 10 *$\mathcal{F}_2$ is closed under $n \mapsto \lceil \sqrt{n} \,\rceil$ if and only if $\mathcal{F}_2 = SPEC$.*

Proof Suppose $\mathcal{F}_2 = SPEC$. Recall that, if we identify natural numbers with their representations in binary, then *SPEC* is the class of languages accepted in exponential time by a non-deterministic Turing machine. Since this complexity class is closed under $n \mapsto \lceil \sqrt{n} \,\rceil$, it follows that *SPEC* is closed under $n \mapsto \lceil \sqrt{n} \,\rceil$. Hence, $\mathcal{F}_2$ is closed under $n \mapsto \lceil \sqrt{n} \,\rceil$.

Suppose that $\mathcal{F}_2$ is closed under $n \mapsto \lceil \sqrt{n} \,\rceil$. Let $S \in SPEC$. Clearly there exists m such that $S \in \mathcal{F}_{2^m}$. By applying Corollary 4 m times, $S^{2^m} \in \mathcal{F}_2$. By the hypothesis applied m times, $S \in \mathcal{F}_2$. □

This theorem is also an easy consequence of Fagin's work in [5].

6 Binary Relations with Bounded Outdegree

Let R be a binary relation symbol. Recall from the introduction that BIN^1_{bo} is the class of spectra of $\{R\}$-sentences where $\{R\}$ has bounded outdegree. In [3], it is shown that BIN^1_{bo} is the class of spectra of sentences involving at most unary function symbols, but no relation is known between BIN^1_{bo} and the class BIN^1 except for the trivial inclusion $\mathit{BIN}^1_{bo} \subseteq \mathit{BIN}^1$. In this section, we prove that, if $\mathit{BIN}^1 = \mathit{BIN}^1_{bo}$, then Fagin's arity hierarchy collapses.

Theorem 11 *If $S \in \mathit{BIN}^1_{bo}$, then $\{n : n^2 \in S\} \in \mathcal{F}_3$.*

Proof Suppose $S \in \mathit{BIN}^1_{bo}$ and let $\hat{S} = S \cap \{n^2 : n \in \omega\}$. It follows from results in [3] and [10] that $\{n^2 : n \in \omega\}$ is in BIN^1_{bo}. So there is a set $\hat{S} \in \mathit{BIN}^1_{bo}$ such that $\hat{S} \subseteq \{n^2 : n \in \omega\}$ and $\{n : n^2 \in \hat{S}\} \in \mathcal{F}_3$ if and only if $\{n : n^2 \in S\} \in \mathcal{F}_3$. Therefore, we can assume that $S \subseteq \{n^2 : n \in \omega\}$, because otherwise we can simply use the set $\hat{S}$.

Suppose $S = \mathit{Sp}(\phi)$ where ϕ involves one binary relation symbol R with outdegree bounded by k. To simplify the notation in this proof, we will use the same symbol to denote a relation symbol and its interpretation. We feel that this will not create confusion.

Let $\mathcal{M} = \langle n \times n, R\rangle$ be an arbitrary structure where R is binary with outdegree k. We construct a new structure $\mathcal{M}^* = \langle n, <, T, A_1, \ldots, A_N\rangle$, where T is ternary, $<$ is the usual linear order on ω, and, for each $i \le N$, A_i is binary. The way in which the value N is determined will be clear below.

Define T as follows:

$$Tabc \iff R(a,b)(c,d) \text{ or } R(a,b)(d,c), \text{for some } d \in n$$

Hence, with each $(a,b) \in n \times n$, T associates the set of coordinates that occur in some pair (c,d) such that $R(a,b)(c,d)$. Note that the set of pairs (c,d) such that $R(a,b)(c,d)$ can not be recovered from T alone.

The relations $A_1, \ldots, A_N$ are defined in such a way that from (a,b) it is possible to recover all pairs (c,d) such that $R(a,b)(c,d)$. There are three key points:

1. since R has outdegree k, $|\{c : Tabc\}| \le 2k$
2. up to isomorphism, there are less than 2^{2k+1} ordered directed graphs on sets of size at most $2k$
3. from $\{c : Tabc\}$ and the isomorphism type $\mathfrak{I}_{(a,b)}$ of the ordered directed graph on $\{c : Tabc\}$ with edge set $\{(c,d) : R(a,b)(c,d)\}$, it is possible to recover all pairs (c,d) such that $R(a,b)(c,d)$.

The relation symbols $A_1, \ldots, A_N$ tell us what $\mathfrak{I}_{(a,b)}$ is for each pair (a,b).

In more detail, define

$$\mathfrak{I}_{(a,b)} = \langle\{c : Tabc\}, R^{(a,b)}, <^\circ\rangle / \cong$$

where $<^\circ$ is the restriction of $<$ and $R^{(a,b)}$ is the binary relation $\{(c,d) : R(a,b)(c,d)\}$. Let $\mathcal{A}_1, \ldots, \mathcal{A}_N$ be an enumeration of pairwise non-isomorphic ordered directed graphs of cardinality at most $2k$ such that every isomorphism type is represented. For each $i \le N$, let $\mathcal{A}_i = \langle n_i, Q_i, <_i\rangle$ where $<_i$ is the usual order on n_i. Define $A_1, \ldots, A_N$ on n as follows:

$$A_i ab \iff \mathcal{A}_i \in \mathfrak{I}_{(a,b)}.$$

For each $i \le N$, let $\lambda_{\mathcal{A}_i}(x,y)$ be a formula over $\{<\}$ such that $\mathcal{A}_i \models \lambda_{\mathcal{A}_i}[a,b]$ if and only if $Q_i ab$. Given an $\{R\}$-formula ψ, let ψ^* be the formula which is obtained from ψ by performing the following steps:

- replace each occurrence of $\exists x$ with $\exists x_1 \exists x_2$
- replace each occurrence of $x = y$ with $x_1 = y_1 \wedge x_2 = y_2$
- replace each occurrence of Rxy with

$$Tx_1x_2y_1 \wedge Tx_1x_2y_2 \wedge \bigvee_{j=1}^{N} (A_j x_1 x_2 \wedge \lambda_{\mathcal{A}_j}(y_1, y_2)).$$

By a simple induction, one can show that, for any formula $\psi(\bar{x})$,

$$\begin{aligned} & \mathcal{M} \models \psi[(a_1,a_2), \ldots, (a_{2p-1}, a_{2p})] \\ \iff\; & \mathcal{M}^* \models \psi^*[a_1, \ldots, a_{2p}]. \end{aligned}$$

We omit the details of the induction.

It is straightforward to show that there is a sentence α whose models are, up to isomorphism, the structures $\mathcal{M}^*$ as $\mathcal{M}$ runs through all structures of the form $\langle n \times n, R\rangle$. The sentence α will be the conjunction of a number of sentences. For example, one conjunct will assert that the relations $A_1, \ldots, A_N$ partition the ordered pairs of elements of the universe; another will assert that T links each pair of elements of the universe to at most $2k$ points. Let ϕ_1 be the conjunction of α with ϕ^*.

We show that $\mathit{Sp}(\phi_1) = \{n : n^2 \in S\}$. Suppose $m \in \{n : n^2 \in S\}$. So there is an $\mathcal{M} = \langle m \times m, R\rangle$ such that $\mathcal{M} \models \phi$. By the induction, it follows that $\mathcal{M}^* \models \phi^*$. It is clear from the construction that $\mathcal{M}^* \models \phi_1$.

Suppose that $m \in \mathit{Sp}(\phi_1)$, so there exists $\mathcal{A}$ with $|A| = m$ such that $\mathcal{A} \models \phi^* \wedge \alpha$. Since $\mathcal{A} \models \alpha$, $\mathcal{A} \cong \mathcal{M}^*$ for some $\mathcal{M} = \langle m \times m, R\rangle$. By the induction, $\mathcal{M} \models \phi$ and, hence, $m^2 \in \mathit{Sp}(\phi)$. □

Notice that there is no obvious way in which the method used to prove Theorem 11 can be extended to the unbounded outdegree case.

As a consequence of Theorem 11, we get another sufficient condition for the collapse of Fagin's hierarchy

Theorem 12 *If $BIN^1_{bo} = \mathcal{F}_2$, then $(\forall k \geq 3)\ \mathcal{F}_k = \mathcal{F}_3$.*

Proof Assume $BIN^1_{bo} = \mathcal{F}_2$ and suppose $S \in \mathcal{F}_4$. By Theorem 7, $S^2 \in \mathcal{F}_2 = BIN^1_{bo}$. Since $\{n : n^2 \in S^2\} = S$, it follows from Theorem 11 that $S \in \mathcal{F}_3$. Hence $\mathcal{F}_4 \subseteq \mathcal{F}_3$. It is easy to see that $\mathcal{F}_3 \subseteq \mathcal{F}_4$ and, therefore, $\mathcal{F}_3 = \mathcal{F}_4$. In [5], Fagin proves that, if $\mathcal{F}_p = \mathcal{F}_{p+1}$ for some p, then $\mathcal{F}_p = \mathcal{F}_k$ for all $k \geq p$. This completes the proof. □

We can prove a similar result involving only the classes BIN^1 and BIN^1_{bo}, but we need the following lemma.

Lemma 2 *$\mathcal{F}_2 = BIN^1_{bo}$ if and only if $BIN^1 = BIN^1_{bo}$.*

Proof If $\mathcal{F}_2 = BIN^1_{bo}$, then clearly $BIN^1 = BIN^1_{bo}$.

Conversely, assume $BIN^1 = BIN^1_{bo}$. Let $\mathcal{F}_{2,bo}$ be the obvious restriction of $\mathcal{F}_2$. By looking at the proof of Theorem 4, it is clear that $\mathcal{F}_{2,bo}$ is closed under $n \mapsto \lceil \frac{n}{2} \rceil$. Moreover, using techniques employed in [3] it can be showen that $\mathcal{F}_{2,bo} = BIN^1_{bo}$. Hence BIN^1_{bo} is closed under $n \mapsto \lceil \frac{n}{2} \rceil$. It follows by our assumption that BIN^1 is closed under $n \mapsto \lceil \frac{n}{2} \rceil$ and, therefore, $BIN^1 = \mathcal{F}_2$ by Theorem 6. Therefore $BIN^1_{bo} = \mathcal{F}_2$. □

Theorem 13 *If $BIN^1_{bo} = BIN^1$, then $(\forall k \geq 3)\ \mathcal{F}_k = \mathcal{F}_3$.*

Proof Immediate. □

7. Conclusion

In [12], More asks if BIN^1 is closed under any subdiagonal function. Our results show that More's problem is harder than it first appears. Determing if any spectrum is outside BIN^1 has proven to be very difficult, and we have shown that proving closure under many natural subdiagonal functions is equally difficult. Hence, although we have not solved More's problem, we have given some evidence that finding a solution may be quite a challenge.

There are some natural questions that arise from the results we present. First of all, two of our results may not be best possible. In Theorem 3, it is shown that $\mathcal{F}_2^{1,2} = BIN^1$ if and only if BIN^1 is closed under $n \mapsto n - 1$. It is natural to ask whether or not $\mathcal{F}_2^{1,2}$ can be replaced by $\mathcal{F}_2^{1,1}$. In §6, we give a condition which implies that Fagin's hierarchy collapses to $\mathcal{F}_3$. This result would be more satisfying if it could be modified to get a collapse to $\mathcal{F}_2$.

In order to develop more powerful model-theoretic techniques, it would be valuable to revisit some established results in the theory of spectra. For example, Grandjean uses a complexity theoretic argument to prove that there is a strict hierarchy of spectrum classes based on the number of universal quantifiers allowed in the sentence. It would be nice to give a semantic proof of this result. Another result worth revisiting is Fagin's proof that *SPEC* is not closed under $\log_2$ [4]. We are currently preparing a paper that provides a new semantic proof of this result.

In this paper, we have taken the position that the natural way to study spectra is from the perspective of model theory. This is the position taken by More [12], and it constitutes a return to the historical roots of Asser's problem. However, in order to make any significant progress on the long-standing open problems in the theory of spectra, we need to develop more sophisticated tools.

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