BCIT Physics 8400 Textbook

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5 | RELATIVITY

Figure 5.1 Special relativity explains how time passes slightly differently on Earth and within the rapidly moving global positioning satellite (GPS). GPS units in vehicles could not find their correct location on Earth without taking this correction into account. (credit: USAF)

Chapter Outline

Introduction

The special theory of relativity was proposed in 1905 by Albert Einstein (1879–1955). It describes how time, space, and physical phenomena appear in different frames of reference that are moving at constant velocity with respect to each other. This differs from Einstein's later work on general relativity, which deals with any frame of reference, including accelerated frames.

The theory of relativity led to a profound change in the way we perceive space and time. The "common sense" rules that we use to relate space and time measurements in the Newtonian worldview differ seriously from the correct rules at speeds near the speed of light. For example, the special theory of relativity tells us that measurements of length and time intervals are not the same in reference frames moving relative to one another. A particle might be observed to have a lifetime of 1.0×10^{-8} s in one reference frame, but a lifetime of 2.0×10^{-8} s in another; and an object might be measured to be 2.0 m long in one frame and 3.0 m long in another frame. These effects are usually significant only at speeds comparable to the speed of light, but even at the much lower speeds of the global positioning satellite, which requires extremely accurate time measurements to function, the different lengths of the same distance in different frames of reference are significant enough that they need to be taken into account.

Unlike Newtonian mechanics, which describes the motion of particles, or Maxwell's equations, which specify how the electromagnetic field behaves, special relativity is not restricted to a particular type of phenomenon. Instead, its rules on space and time affect all fundamental physical theories.

The modifications of Newtonian mechanics in special relativity do not invalidate classical Newtonian mechanics or require its replacement. Instead, the equations of relativistic mechanics differ meaningfully from those of classical Newtonian mechanics only for objects moving at relativistic speeds (i.e., speeds less than, but comparable to, the speed of light). In the macroscopic world that you encounter in your daily life, the relativistic equations reduce to classical equations, and the predictions of classical Newtonian mechanics agree closely enough with experimental results to disregard relativistic corrections.

5.1 | Invariance of Physical Laws

Learning Objectives

By the end of this section, you will be able to:

- Describe the theoretical and experimental issues that Einstein's theory of special relativity addressed.
- State the two postulates of the special theory of relativity.

Suppose you calculate the hypotenuse of a right triangle given the base angles and adjacent sides. Whether you calculate the hypotenuse from one of the sides and the cosine of the base angle, or from the Pythagorean theorem, the results should agree. Predictions based on different principles of physics must also agree, whether we consider them principles of mechanics or principles of electromagnetism.

Albert Einstein pondered a disagreement between predictions based on electromagnetism and on assumptions made in classical mechanics. Specifically, suppose an observer measures the velocity of a light pulse in the observer's own **rest frame**; that is, in the frame of reference in which the observer is at rest. According to the assumptions long considered obvious in classical mechanics, if an observer measures a velocity $\vec{\mathbf{v}}$ in one frame of reference, and that frame of reference is moving with velocity $\vec{\mathbf{u}}$ past a second reference frame, an observer in the second frame measures the original velocity as $\vec{\mathbf{v}}' = \vec{\mathbf{v}} + \vec{\mathbf{u}}$. This sum of velocities is often referred to as **Galilean relativity**. If this principle is correct, the pulse of light that the observer measures as traveling with speed c travels at speed $c + u$ measured in the frame of the second observer. If we reasonably assume that the laws of electrodynamics are the same in both frames of reference, then the predicted speed of light (in vacuum) in both frames should be $c = 1/\sqrt{\varepsilon_0 \mu_0}$. Each observer should measure the same speed of the light pulse with respect to that observer's own rest frame. To reconcile difficulties of this kind, Einstein constructed his **special theory of relativity**, which introduced radical new ideas about time and space that have since been confirmed experimentally.

Inertial Frames

All velocities are measured relative to some frame of reference. For example, a car's motion is measured relative to its starting position on the road it travels on; a projectile's motion is measured relative to the surface from which it is launched; and a planet's orbital motion is measured relative to the star it orbits. The frames of reference in which mechanics takes the simplest form are those that are not accelerating. Newton's first law, the law of inertia, holds exactly in such a frame.

Inertial Reference Frame

An **inertial frame of reference** is a reference frame in which a body at rest remains at rest and a body in motion moves at a constant speed in a straight line unless acted upon by an outside force.

For example, to a passenger inside a plane flying at constant speed and constant altitude, physics seems to work exactly the same as when the passenger is standing on the surface of Earth. When the plane is taking off, however, matters are somewhat more complicated. In this case, the passenger at rest inside the plane concludes that a net force F on an object is not equal to the product of mass and acceleration, *ma*. Instead, *F* is equal to *ma* plus a fictitious force. This situation is not as simple as in an inertial frame. The term “special” in “special relativity” refers to dealing only with inertial frames of reference. Einstein’s later theory of general relativity deals with all kinds of reference frames, including accelerating, and therefore non-inertial, reference frames.

Einstein’s First Postulate

Not only are the principles of classical mechanics simplest in inertial frames, but they are the same in all inertial frames. Einstein based the **first postulate** of his theory on the idea that this is true for all the laws of physics, not merely those in mechanics.

First Postulate of Special Relativity

The laws of physics are the same in all inertial frames of reference.

This postulate denies the existence of a special or preferred inertial frame. The laws of nature do not give us a way to endow any one inertial frame with special properties. For example, we cannot identify any inertial frame as being in a state of “absolute rest.” We can only determine the relative motion of one frame with respect to another.

There is, however, more to this postulate than meets the eye. The laws of physics include only those that satisfy this postulate. We will see that the definitions of energy and momentum must be altered to fit this postulate. Another outcome of this postulate is the famous equation $E = mc^2$, which relates energy to mass.

Einstein’s Second Postulate

The second postulate upon which Einstein based his theory of special relativity deals with the speed of light. Late in the nineteenth century, the major tenets of classical physics were well established. Two of the most important were the laws of electromagnetism and Newton’s laws. Investigations such as Young’s double-slit experiment in the early 1800s had convincingly demonstrated that light is a wave. Maxwell’s equations of electromagnetism implied that electromagnetic waves travel at $c = 3.00\times10^8$ m/s in a vacuum, but they do not specify the frame of reference in which light has this speed. Many types of waves were known, and all travelled in some medium. Scientists therefore assumed that some medium carried the light, even in a vacuum, and that light travels at a speed *c* relative to that medium (often called “the aether”).

Starting in the mid-1880s, the American physicist A.A. Michelson, later aided by E.W. Morley, made a series of direct measurements of the speed of light. They intended to deduce from their data the speed *v* at which Earth was moving through the mysterious medium for light waves. The speed of light measured on Earth should have been $c + v$ when Earth’s motion was opposite to the medium’s flow at speed *u* past the Earth, and $c - v$ when Earth was moving in the same direction as the medium. The results of their measurements were startling.

Michelson-Morley Experiment

The **Michelson-Morley experiment** demonstrated that the speed of light in a vacuum is independent of the motion of Earth about the Sun.

The eventual conclusion derived from this result is that light, unlike mechanical waves such as sound, does not need a medium to carry it. Furthermore, the Michelson-Morley results implied that the speed of light *c* is independent of the motion of the source relative to the observer. That is, everyone observes light to move at speed *c* regardless of how they move relative to the light source or to one another. For several years, many scientists tried unsuccessfully to explain these results within the framework of Newton’s laws.

In addition, there was a contradiction between the principles of electromagnetism and the assumption made in Newton’s laws about relative velocity. Classically, the velocity of an object in one frame of reference and the velocity of that object in a second frame of reference relative to the first should combine like simple vectors to give the velocity seen in the second frame. If that were correct, then two observers moving at different speeds would see light traveling at different speeds. Imagine what a light wave would look like to a person traveling along with it (in vacuum) at a speed *c*. If such a motion were possible, then the wave would be stationary relative to the observer. It would have electric and magnetic fields whose strengths varied with position but were constant in time. This is not allowed by Maxwell’s equations. So either Maxwell’s equations are different in different inertial frames, or an object with mass cannot travel at speed *c*. Einstein concluded that the latter is true: An object with mass cannot travel at speed c. Maxwell's equations are correct, but Newton's addition of velocities is not correct for light.

Not until 1905, when Einstein published his first paper on special relativity, was the currently accepted conclusion reached. Based mostly on his analysis that the laws of electricity and magnetism would not allow another speed for light, and only slightly aware of the Michelson-Morley experiment, Einstein detailed his **second postulate of special relativity**.

Second Postulate of Special Relativity

Light travels in a vacuum with the same speed c in any direction in all inertial frames.

In other words, the speed of light has the same definite speed for any observer, regardless of the relative motion of the source. This deceptively simple and counterintuitive postulate, along with the first postulate, leave all else open for change. Among the changes are the loss of agreement on the time between events, the variation of distance with speed, and the realization that matter and energy can be converted into one another. We describe these concepts in the following sections.

5.1 Check Your Understanding Explain how special relativity differs from general relativity.

5.2 | Relativity of Simultaneity

Learning Objectives

By the end of this section, you will be able to:

- Show from Einstein's postulates that two events measured as simultaneous in one inertial frame are not necessarily simultaneous in all inertial frames.
- Describe how simultaneity is a relative concept for observers in different inertial frames in relative motion.

Do time intervals depend on who observes them? Intuitively, it seems that the time for a process, such as the elapsed time for a foot race (**Figure 5.2**), should be the same for all observers. In everyday experiences, disagreements over elapsed time have to do with the accuracy of measuring time. No one would be likely to argue that the actual time interval was different for the moving runner and for the stationary clock displayed. Carefully considering just how time is measured, however, shows that elapsed time does depends on the relative motion of an observer with respect to the process being measured.

Figure 5.2 Elapsed time for a foot race is the same for all observers, but at relativistic speeds, elapsed time depends on the motion of the observer relative to the location where the process being timed occurs. (credit: "Jason Edward Scott Bain"/Flickr)

Consider how we measure elapsed time. If we use a stopwatch, for example, how do we know when to start and stop the watch? One method is to use the arrival of light from the event. For example, if you're in a moving car and observe the light arriving from a traffic signal change from green to red, you know it's time to step on the brake pedal. The timing is more accurate if some sort of electronic detection is used, avoiding human reaction times and other complications.

Now suppose two observers use this method to measure the time interval between two flashes of light from flash lamps that are a distance apart (**Figure 5.3**). An observer *A* is seated midway on a rail car with two flash lamps at opposite sides equidistant from her. A pulse of light is emitted from each flash lamp and moves toward observer *A*, shown in frame (a) of the figure. The rail car is moving rapidly in the direction indicated by the velocity vector in the diagram. An observer *B* standing on the platform is facing the rail car as it passes and observes both flashes of light reach him simultaneously, as shown in frame (c). He measures the distances from where he saw the pulses originate, finds them equal, and concludes that the pulses were emitted simultaneously.

However, because of Observer *A*'s motion, the pulse from the right of the railcar, from the direction the car is moving, reaches her before the pulse from the left, as shown in frame (b). She also measures the distances from within her frame of reference, finds them equal, and concludes that the pulses were not emitted simultaneously.

The two observers reach conflicting conclusions about whether the two events at well-separated locations were simultaneous. Both frames of reference are valid, and both conclusions are valid. Whether two events at separate locations are simultaneous depends on the motion of the observer relative to the locations of the events.

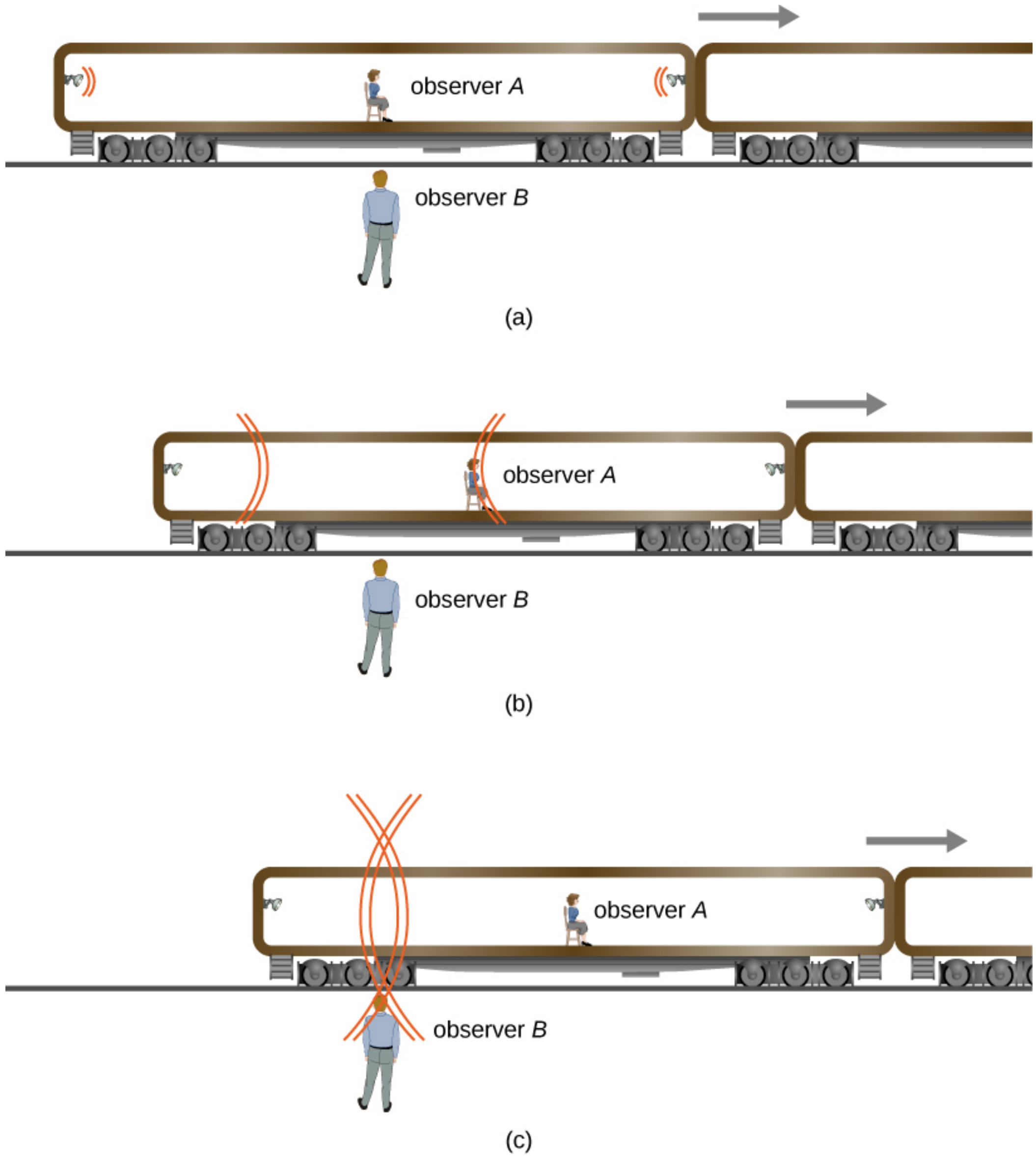


Figure 5.3 (a) Two pulses of light are emitted simultaneously relative to observer *B*. (c) The pulses reach observer *B*'s position simultaneously. (b) Because of *A*'s motion, she sees the pulse from the right first and concludes the bulbs did not flash simultaneously. Both conclusions are correct.

Here, the relative velocity between observers affects whether two events a distance apart are observed to be simultaneous. *Simultaneity is not absolute*. We might have guessed (incorrectly) that if light is emitted simultaneously, then two observers halfway between the sources would see the flashes simultaneously. But careful analysis shows this cannot be the case if the speed of light is the same in all inertial frames.

This type of *thought experiment* (in German, "Gedankenexperiment") shows that seemingly obvious conclusions must be changed to agree with the postulates of relativity. The validity of thought experiments can only be determined by actual observation, and careful experiments have repeatedly confirmed Einstein's theory of relativity.

5.3 | Time Dilation

Learning Objectives

By the end of this section, you will be able to:

- Explain how time intervals can be measured differently in different reference frames.
- Describe how to distinguish a proper time interval from a dilated time interval.
- Describe the significance of the muon experiment.
- Explain why the twin paradox is not a contradiction.
- Calculate time dilation given the speed of an object in a given frame.

The analysis of simultaneity shows that Einstein's postulates imply an important effect: Time intervals have different values when measured in different inertial frames. Suppose, for example, an astronaut measures the time it takes for a pulse of light to travel a distance perpendicular to the direction of his ship's motion (relative to an earthbound observer), bounce off a mirror, and return (**Figure 5.4**). How does the elapsed time that the astronaut measures in the spacecraft compare with the elapsed time that an earthbound observer measures by observing what is happening in the spacecraft?

Examining this question leads to a profound result. The elapsed time for a process depends on which observer is measuring it. In this case, the time measured by the astronaut (within the spaceship where the astronaut is at rest) is smaller than the time measured by the earthbound observer (to whom the astronaut is moving). The time elapsed for the same process is different for the observers, because the distance the light pulse travels in the astronaut's frame is smaller than in the earthbound frame, as seen in **Figure 5.4**. Light travels at the same speed in each frame, so it takes more time to travel the greater distance in the earthbound frame.

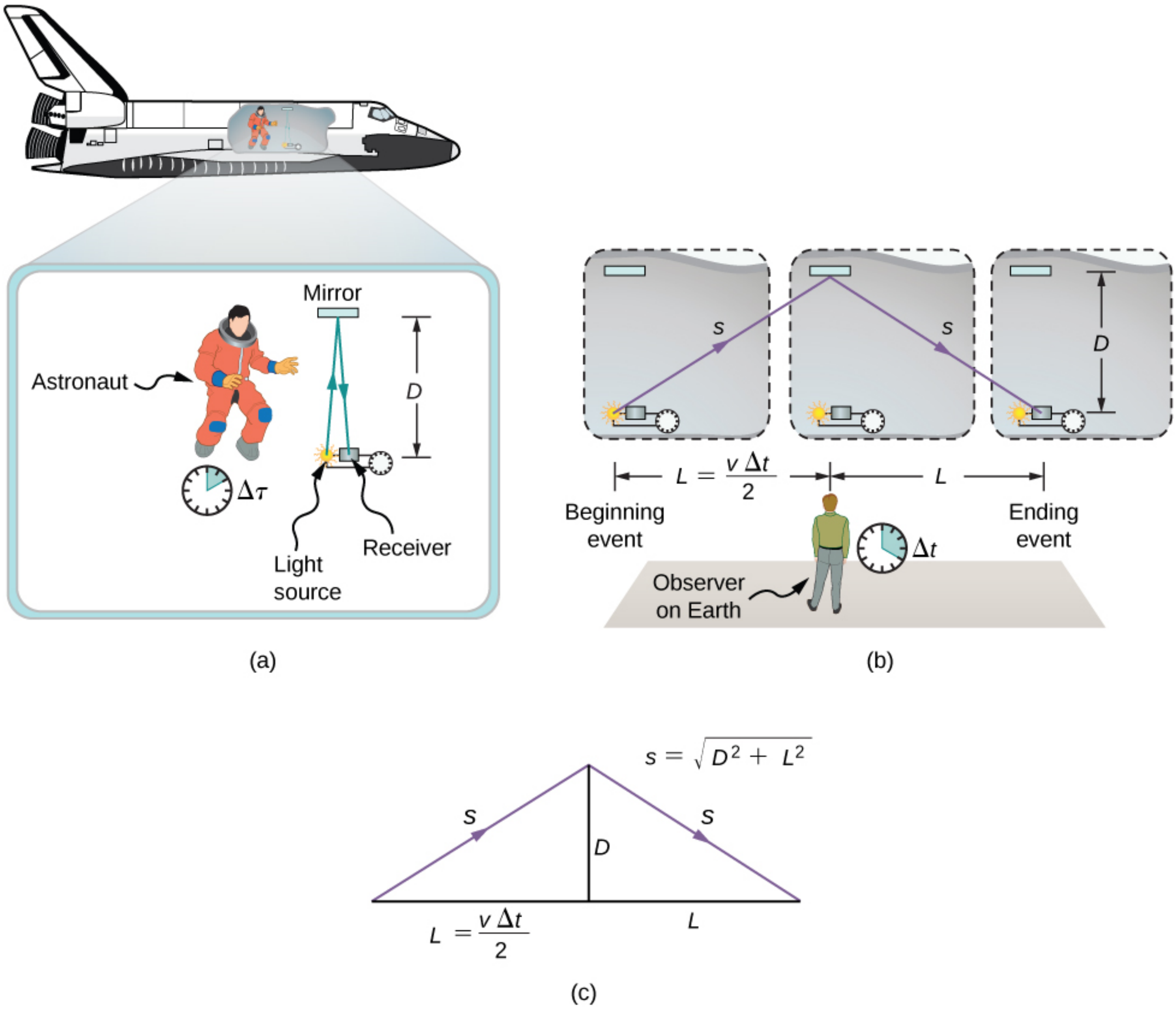


Figure 5.4 (a) An astronaut measures the time $\Delta\tau$ for light to travel distance $2D$ in the astronaut's frame. (b) A NASA scientist on Earth sees the light follow the longer path $2s$ and take a longer time Δt. (c) These triangles are used to find the relationship between the two distances D and s.

Time Dilation

Time dilation is the lengthening of the time interval between two events for an observer in an inertial frame that is moving with respect to the rest frame of the events (in which the events occur at the same location).

To quantitatively compare the time measurements in the two inertial frames, we can relate the distances in **Figure 5.4** to each other, then express each distance in terms of the time of travel (respectively either Δt or $\Delta\tau$) of the pulse in the corresponding reference frame. The resulting equation can then be solved for Δt in terms of $\Delta\tau$.

The lengths D and L in **Figure 5.4** are the sides of a right triangle with hypotenuse s. From the Pythagorean theorem,

$$s^2 = D^2 + L^2.$$

The lengths $2s$ and $2L$ are, respectively, the distances that the pulse of light and the spacecraft travel in time Δt in the earthbound observer's frame. The length D is the distance that the light pulse travels in time $\Delta\tau$ in the astronaut's frame. This gives us three equations:

$$2s = c\Delta t;\ 2L = v\Delta t;\ 2D = c\Delta\tau.$$

Note that we used Einstein's second postulate by taking the speed of light to be *c* in both inertial frames. We substitute these results into the previous expression from the Pythagorean theorem:

$$\begin{aligned} s^2 &= D^2 + L^2 \\ \left(c\frac{\Delta t}{2}\right)^2 &= \left(c\frac{\Delta \tau}{2}\right)^2 + \left(v\frac{\Delta t}{2}\right)^2. \end{aligned}$$

Then we rearrange to obtain

$$(c\Delta t)^2 - (v\Delta t)^2 = (c\Delta \tau)^2.$$

Finally, solving for Δt in terms of $\Delta \tau$ gives us

$$\Delta t = \frac{\Delta \tau}{\sqrt{1 - (v/c)^2}}. \quad (5.1)$$

This is equivalent to

$$\Delta t = \gamma \Delta \tau,$$

where γ is the relativistic factor (often called the Lorentz factor) given by

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (5.2)$$

and *v* and *c* are the speeds of the moving observer and light, respectively.

Note the asymmetry between the two measurements. Only one of them is a measurement of the time interval between two events—the emission and arrival of the light pulse—at the same position. It is a measurement of the time interval in the rest frame of a single clock. The measurement in the earthbound frame involves comparing the time interval between two events that occur at different locations. The time interval between events that occur at a single location has a separate name to distinguish it from the time measured by the earthbound observer, and we use the separate symbol $\Delta \tau$ to refer to it throughout this chapter.

Proper Time

The **proper time** interval $\Delta \tau$ between two events is the time interval measured by an observer for whom both events occur at the same location.

The equation relating Δt and $\Delta \tau$ is truly remarkable. First, as stated earlier, elapsed time is not the same for different observers moving relative to one another, even though both are in inertial frames. A proper time interval $\Delta \tau$ for an observer who, like the astronaut, is moving with the apparatus, is smaller than the time interval for other observers. It is the smallest possible measured time between two events. The earthbound observer sees time intervals within the moving system as dilated (i.e., lengthened) relative to how the observer moving relative to Earth sees them within the moving system. Alternatively, according to the earthbound observer, less time passes between events within the moving frame. Note that the shortest elapsed time between events is in the inertial frame in which the observer sees the events (e.g., the emission and arrival of the light signal) occur at the same point.

This time effect is real and is not caused by inaccurate clocks or improper measurements. Time-interval measurements of the same event differ for observers in relative motion. The dilation of time is an intrinsic property of time itself. All clocks moving relative to an observer, including biological clocks, such as a person's heartbeat, or aging, are observed to run more slowly compared with a clock that is stationary relative to the observer.

Note that if the relative velocity is much less than the speed of light $(v<<c)$, then v^2/c^2 is extremely small, and the elapsed times Δt and $\Delta\tau$ are nearly equal. At low velocities, physics based on modern relativity approaches classical physics—everyday experiences involve very small relativistic effects. However, for speeds near the speed of light, v^2/c^2 is close to one, so $\sqrt{1-v^2/c^2}$ is very small and Δt becomes significantly larger than $\Delta\tau$.

Half-Life of a Muon

There is considerable experimental evidence that the equation $\Delta t = \gamma\Delta\tau$ is correct. One example is found in cosmic ray particles that continuously rain down on Earth from deep space. Some collisions of these particles with nuclei in the upper atmosphere result in short-lived particles called muons. The half-life (amount of time for half of a material to decay) of a muon is 1.52 μs when it is at rest relative to the observer who measures the half-life. This is the proper time interval $\Delta\tau$. This short time allows very few muons to reach Earth's surface and be detected if Newtonian assumptions about time and space were correct. However, muons produced by cosmic ray particles have a range of velocities, with some moving near the speed of light. It has been found that the muon's half-life as measured by an earthbound observer (Δt) varies with velocity exactly as predicted by the equation $\Delta t = \gamma\Delta\tau$. The faster the muon moves, the longer it lives. We on Earth see the muon last much longer than its half-life predicts within its own rest frame. As viewed from our frame, the muon decays more slowly than it does when at rest relative to us. A far larger fraction of muons reach the ground as a result.

Before we present the first example of solving a problem in relativity, we state a strategy you can use as a guideline for these calculations.

Problem-Solving Strategy: Relativity

1. Make a list of what is given or can be inferred from the problem as stated (identify the knowns). Look in particular for information on relative velocity *v*.
2. Identify exactly what needs to be determined in the problem (identify the unknowns).
3. Make certain you understand the conceptual aspects of the problem before making any calculations (express the answer as an equation). Decide, for example, which observer sees time dilated or length contracted before working with the equations or using them to carry out the calculation. If you have thought about who sees what, who is moving with the event being observed, who sees proper time, and so on, you will find it much easier to determine if your calculation is reasonable.
4. Determine the primary type of calculation to be done to find the unknowns identified above (do the calculation). You will find the section summary helpful in determining whether a length contraction, relativistic kinetic energy, or some other concept is involved.

Note *that you should not round off during the calculation*. As noted in the text, you must often perform your calculations to many digits to see the desired effect. You may round off at the very end of the problem solution, but do not use a rounded number in a subsequent calculation. Also, check the answer to see if it is reasonable: Does it make sense? This may be more difficult for relativity, which has few everyday examples to provide experience with what is reasonable. But you can look for velocities greater than *c* or relativistic effects that are in the wrong direction (such as a time contraction where a dilation was expected).

Example 5.1

Time Dilation in a High-Speed Vehicle

The Hypersonic Technology Vehicle 2 (HTV-2) is an experimental rocket vehicle capable of traveling at 21,000 km/h (5830 m/s). If an electronic clock in the HTV-2 measures a time interval of exactly 1-s duration, what would observers on Earth measure the time interval to be?

Strategy

Apply the time dilation formula to relate the proper time interval of the signal in HTV-2 to the time interval measured on the ground.

Solution

a. Identify the knowns: $\Delta\tau = 1\text{ s}; v = 5830\text{ m/s}$.

b. Identify the unknown: Δt.

c. Express the answer as an equation:

$$\Delta t = \gamma\Delta\tau = \frac{\Delta\tau}{\sqrt{1-\frac{v^2}{c^2}}}.$$

d. Do the calculation. Use the expression for γ to determine Δt from $\Delta\tau$:

$$\begin{aligned}\Delta t &= \frac{1\text{ s}}{\sqrt{1-\left(\frac{5830\text{ m/s}}{3.00\times10^8\text{ m/s}}\right)^2}}\\ &= 1.000000000189\text{ s}\\ &= 1\text{ s} + 1.89\times10^{-10}\text{ s.}\end{aligned}$$

Significance

The very high speed of the HTV-2 is still only 10^{-5} times the speed of light. Relativistic effects for the HTV-2 are negligible for almost all purposes, but are not zero.

Example 5.2

What Speeds are Relativistic?

How fast must a vehicle travel for 1 second of time measured on a passenger's watch in the vehicle to differ by 1% for an observer measuring it from the ground outside?

Strategy

Use the time dilation formula to find *v/c* for the given ratio of times.

Solution

a. Identify the known:

$$\frac{\Delta\tau}{\Delta t} = \frac{1}{1.01}.$$

b. Identify the unknown: *v/c*.

c. Express the answer as an equation:

$$\begin{aligned} \Delta t &= \gamma \Delta \tau = \frac{1}{\sqrt{1 - v^2/c^2}} \Delta \tau \\ \frac{\Delta \tau}{\Delta t} &= \sqrt{1 - v^2/c^2} \\ \left(\frac{\Delta \tau}{\Delta t}\right)^2 &= 1 - \frac{v^2}{c^2} \\ \frac{v}{c} &= \sqrt{1 - (\Delta \tau / \Delta t)^2}. \end{aligned}$$

d. Do the calculation:

$$\begin{aligned} \frac{v}{c} &= \sqrt{1 - (1/1.01)^2} \\ &= 0.14. \end{aligned}$$

Significance

The result shows that an object must travel at very roughly 10% of the speed of light for its motion to produce significant relativistic time dilation effects.

Example 5.3

Calculating Δt for a Relativistic Event

Suppose a cosmic ray colliding with a nucleus in Earth's upper atmosphere produces a muon that has a velocity $v = 0.950c$. The muon then travels at constant velocity and lives 2.20 μs as measured in the muon's frame of reference. (You can imagine this as the muon's internal clock.) How long does the muon live as measured by an earthbound observer (**Figure 5.5**)?

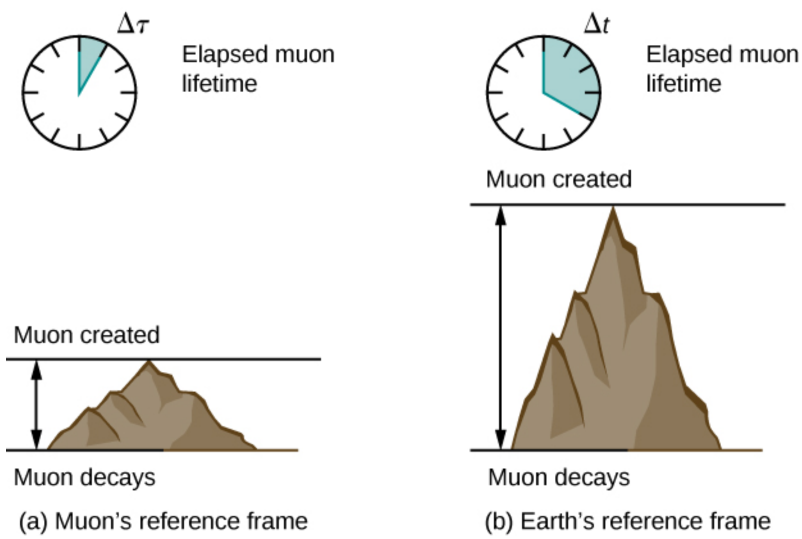


Figure 5.5 A muon in Earth's atmosphere lives longer as measured by an earthbound observer than as measured by the muon's internal clock.

As we will discuss later, in the muon's reference frame, it travels a shorter distance than measured in Earth's reference frame.

Strategy

A clock moving with the muon measures the proper time of its decay process, so the time we are given is $\Delta \tau = 2.20 \mu s$. The earthbound observer measures Δt as given by the equation $\Delta t = \gamma \Delta \tau$. Because the velocity is given, we can calculate the time in Earth's frame of reference.

Solution

a. Identify the knowns: $v = 0.950c,\ \Delta\tau = 2.20\mu\text{s}$.

b. Identify the unknown: Δt.

c. Express the answer as an equation. Use:

$$\Delta t = \gamma\Delta\tau$$

with

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

d. Do the calculation. Use the expression for γ to determine Δt from $\Delta\tau$:

$$\begin{aligned}\Delta t &= \gamma\Delta\tau \\ &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\Delta\tau \\ &= \frac{2.20\mu\text{s}}{\sqrt{1 - (0.950)^2}} \\ &= 7.05\ \mu\text{s}.\end{aligned}$$

Remember to keep extra significant figures until the final answer.

Significance

One implication of this example is that because $\gamma = 3.20$ at 95.0% of the speed of light $(v = 0.950c)$, the relativistic effects are significant. The two time intervals differ by a factor of 3.20, when classically they would be the same. Something moving at 0.950*c* is said to be highly relativistic.

Example 5.4

Relativistic Television

A non-flat screen, older-style television display (**Figure 5.6**) works by accelerating electrons over a short distance to relativistic speed, and then using electromagnetic fields to control where the electron beam strikes a fluorescent layer at the front of the tube. Suppose the electrons travel at 6.00×10^7 m/s through a distance of 0.200 m from the start of the beam to the screen. (a) What is the time of travel of an electron in the rest frame of the television set? (b) What is the electron's time of travel in its own rest frame?

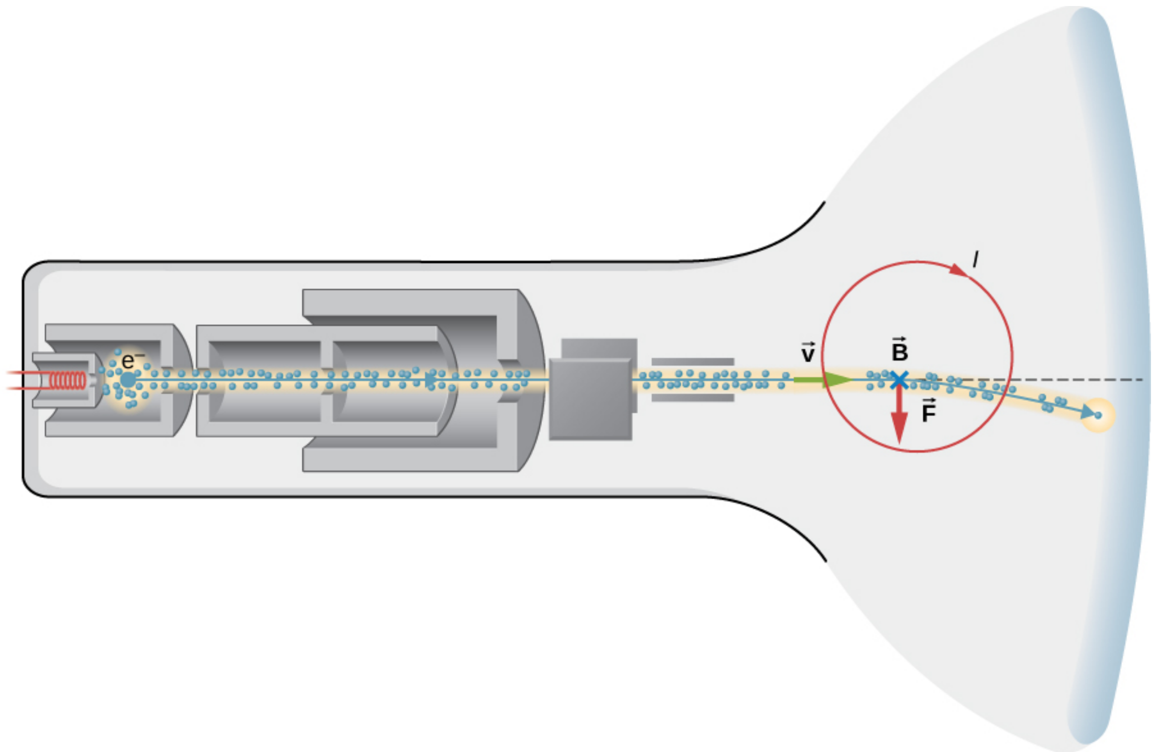


Figure 5.6 The electron beam in a cathode ray tube television display.

Strategy for (a)

(a) Calculate the time from $vt = d$. Even though the speed is relativistic, the calculation is entirely in one frame of reference, and relativity is therefore not involved.

Solution

a. Identify the knowns:

$$v = 6.00 \times 10^7 \text{ m/s}; d = 0.200 \text{ m}.$$

b. Identify the unknown: the time of travel Δt.

c. Express the answer as an equation:

$$\Delta t = \frac{d}{v}.$$

d. Do the calculation:

$$\begin{aligned} t &= \frac{0.200 \text{ m}}{6.00 \times 10^7 \text{ m/s}} \\ &= 3.33 \times 10^{-9} \text{ s}. \end{aligned}$$

Significance

The time of travel is extremely short, as expected. Because the calculation is entirely within a single frame of reference, relativity is not involved, even though the electron speed is close to c.

Strategy for (b)

(b) In the frame of reference of the electron, the vacuum tube is moving and the electron is stationary. The electron-emitting cathode leaves the electron and the front of the vacuum tube strikes the electron with the electron at the same location. Therefore we use the time dilation formula to relate the proper time in the electron rest frame to the time in the television frame.

Solution

a. Identify the knowns (from part a):

$$\Delta t = 3.33 \times 10^{-9}\ \text{s};\ v = 6.00 \times 10^{7}\ \text{m/s};\ d = 0.200\ \text{m}.$$

b. Identify the unknown: τ.

c. Express the answer as an equation:

$$\begin{aligned}\Delta t &= \gamma \Delta \tau = \frac{\Delta \tau}{\sqrt{1 - v^2/c^2}} \\ \Delta \tau &= \Delta t \sqrt{1 - v^2/c^2}.\end{aligned}$$

d. Do the calculation:

$$\begin{aligned}\Delta \tau &= \left(3.33 \times 10^{-9}\ \text{s}\right)\sqrt{1 - \left(\frac{6.00 \times 10^{7}\ \text{m/s}}{3.00 \times 10^{8}\ \text{m/s}}\right)^2} \\ &= 3.26 \times 10^{-9}\ \text{s}.\end{aligned}$$

Significance

The time of travel is shorter in the electron frame of reference. Because the problem requires finding the time interval measured in different reference frames for the same process, relativity is involved. If we had tried to calculate the time in the electron rest frame by simply dividing the 0.200 m by the speed, the result would be slightly incorrect because of the relativistic speed of the electron.

5.2 Check Your Understanding What is γ if $v = 0.650c$?

The Twin Paradox

An intriguing consequence of time dilation is that a space traveler moving at a high velocity relative to Earth would age less than the astronaut's earthbound twin. This is often known as the twin paradox. Imagine the astronaut moving at such a velocity that $\gamma = 30.0$, as in **Figure 5.7**. A trip that takes 2.00 years in her frame would take 60.0 years in the earthbound twin's frame. Suppose the astronaut travels 1.00 year to another star system, briefly explores the area, and then travels 1.00 year back. An astronaut who was 40 years old at the start of the trip would be would be 42 when the spaceship returns. Everything on Earth, however, would have aged 60.0 years. The earthbound twin, if still alive, would be 100 years old.

The situation would seem different to the astronaut in **Figure 5.7**. Because motion is relative, the spaceship would seem to be stationary and Earth would appear to move. (This is the sensation you have when flying in a jet.) Looking out the window of the spaceship, the astronaut would see time slow down on Earth by a factor of $\gamma = 30.0$. Seen from the spaceship, the earthbound sibling will have aged only 2/30, or 0.07, of a year, whereas the astronaut would have aged 2.00 years.

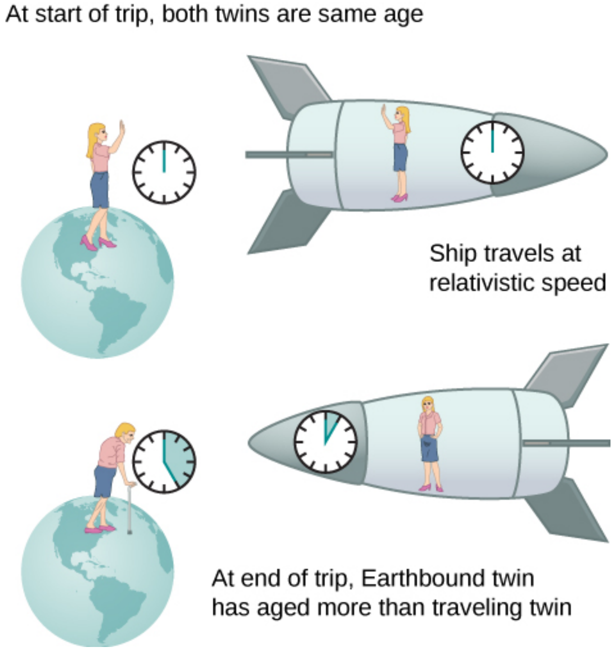


Figure 5.7 The twin paradox consists of the conflicting conclusions about which twin ages more as a result of a long space journey at relativistic speed.

The paradox here is that the two twins cannot both be correct. As with all paradoxes, conflicting conclusions come from a false premise. In fact, the astronaut's motion is significantly different from that of the earthbound twin. The astronaut accelerates to a high velocity and then decelerates to view the star system. To return to Earth, she again accelerates and decelerates. The spacecraft is not in a single inertial frame to which the time dilation formula can be directly applied. That is, the astronaut twin changes inertial references. The earthbound twin does not experience these accelerations and remains in the same inertial frame. Thus, the situation is not symmetric, and it is incorrect to claim that the astronaut observes the same effects as her twin. The lack of symmetry between the twins will be still more evident when we analyze the journey later in this chapter in terms of the path the astronaut follows through four-dimensional space-time.

In 1971, American physicists Joseph Hafele and Richard Keating verified time dilation at low relative velocities by flying extremely accurate atomic clocks around the world on commercial aircraft. They measured elapsed time to an accuracy of a few nanoseconds and compared it with the time measured by clocks left behind. Hafele and Keating's results were within experimental uncertainties of the predictions of relativity. Both special and general relativity had to be taken into account, because gravity and accelerations were involved as well as relative motion.

5.3 **Check Your Understanding** a. A particle travels at $1.90 \times 10^8\,\text{m/s}$ and lives $2.10 \times 10^{-8}\,\text{s}$ when at rest relative to an observer. How long does the particle live as viewed in the laboratory?

b. Spacecraft *A* and *B* pass in opposite directions at a relative speed of $4.00 \times 10^7\,\text{m/s}.$ An internal clock in spacecraft *A* causes it to emit a radio signal for 1.00 s. The computer in spacecraft *B* corrects for the beginning and end of the signal having traveled different distances, to calculate the time interval during which ship *A* was emitting the signal. What is the time interval that the computer in spacecraft *B* calculates?

5.4 | Length Contraction

Learning Objectives

By the end of this section, you will be able to:

- Explain how simultaneity and length contraction are related.
- Describe the relation between length contraction and time dilation and use it to derive the length-contraction equation.

The length of the train car in **Figure 5.8** is the same for all the passengers. All of them would agree on the simultaneous location of the two ends of the car and obtain the same result for the distance between them. But simultaneous events in one inertial frame need not be simultaneous in another. If the train could travel at relativistic speeds, an observer on the ground would see the simultaneous locations of the two endpoints of the car at a different distance apart than observers inside the car. Measured distances need not be the same for different observers when relativistic speeds are involved.

Figure 5.8 People might describe distances differently, but at relativistic speeds, the distances really are different. (credit: “russavia”/Flickr)

Proper Length

Two observers passing each other always see the same value of their relative speed. Even though time dilation implies that the train passenger and the observer standing alongside the tracks measure different times for the train to pass, they still agree that relative speed, which is distance divided by elapsed time, is the same. If an observer on the ground and one on the train measure a different time for the length of the train to pass the ground observer, agreeing on their relative speed means they must also see different distances traveled.

The muon discussed in **Example 5.3** illustrates this concept (**Figure 5.9**). To an observer on Earth, the muon travels at $0.950c$ for 7.05 μs from the time it is produced until it decays. Therefore, it travels a distance relative to Earth of:

$$L_0 = v\Delta t = (0.950)(3.00 \times 10^8 \text{ m/s})(7.05 \times 10^{-6} \text{ s}) = 2.01 \text{ km}.$$

In the muon frame, the lifetime of the muon is 2.20 μs. In this frame of reference, the Earth, air, and ground have only enough time to travel:

$$L = v\Delta\tau = (0.950)(3.00 \times 10^8 \text{ m/s})(2.20 \times 10^{-6} \text{ s}) \text{ km} = 0.627 \text{ km}.$$

The distance between the same two events (production and decay of a muon) depends on who measures it and how they are moving relative to it.

Proper Length

Proper length L_0 is the distance between two points measured by an observer who is at rest relative to both of the points.

The earthbound observer measures the proper length L_0 because the points at which the muon is produced and decays are stationary relative to Earth. To the muon, Earth, air, and clouds are moving, so the distance L it sees is not the proper length.

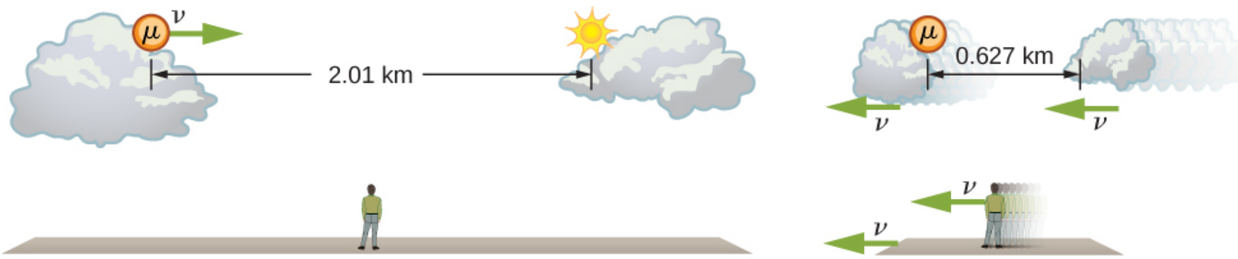


Figure 5.9 (a) The earthbound observer sees the muon travel 2.01 km. (b) The same path has length 0.627 km seen from the muon's frame of reference. The Earth, air, and clouds are moving relative to the muon in its frame, and have smaller lengths along the direction of travel.

Length Contraction

To relate distances measured by different observers, note that the velocity relative to the earthbound observer in our muon example is given by

$$v = \frac{L_0}{\Delta t}.$$

The time relative to the earthbound observer is Δt, because the object being timed is moving relative to this observer. The velocity relative to the moving observer is given by

$$v = \frac{L}{\Delta \tau}.$$

The moving observer travels with the muon and therefore observes the proper time $\Delta \tau$. The two velocities are identical; thus,

$$\frac{L_0}{\Delta t} = \frac{L}{\Delta \tau}.$$

We know that $\Delta t = \gamma \Delta \tau$. Substituting this equation into the relationship above gives

$$L = \frac{L_0}{\gamma}. \quad (5.3)$$

Substituting for γ gives an equation relating the distances measured by different observers.

Length Contraction

Length contraction is the decrease in the measured length of an object from its proper length when measured in a reference frame that is moving with respect to the object:

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}} \quad (5.4)$$

where L_0 is the length of the object in its rest frame, and L is the length in the frame moving with velocity v.

If we measure the length of anything moving relative to our frame, we find its length L to be smaller than the proper length L_0 that would be measured if the object were stationary. For example, in the muon's rest frame, the distance Earth moves between where the muon was produced and where it decayed is shorter than the distance traveled as seen from the Earth's frame. Those points are fixed relative to Earth but are moving relative to the muon. Clouds and other objects are also contracted along the direction of motion as seen from muon's rest frame.

Thus, two observers measure different distances along their direction of relative motion, depending on which one is measuring distances between objects at rest.

But what about distances measured in a direction perpendicular to the relative motion? Imagine two observers moving along their *x*-axes and passing each other while holding meter sticks vertically in the *y*-direction. **Figure 5.10** shows two meter sticks M and M' that are at rest in the reference frames of two boys S and S', respectively. A small paintbrush is attached to the top (the 100-cm mark) of stick M'. Suppose that S' is moving to the right at a very high speed *v* relative to S, and the sticks are oriented so that they are perpendicular, or transverse, to their relative velocity vector. The sticks are held so that as they pass each other, their lower ends (the 0-cm marks) coincide. Assume that when S looks at his stick M afterwards, he finds a line painted on it, just below the top of the stick. Because the brush is attached to the top of the other boy's stick M', S can only conclude that stick M' is less than 1.0 m long.

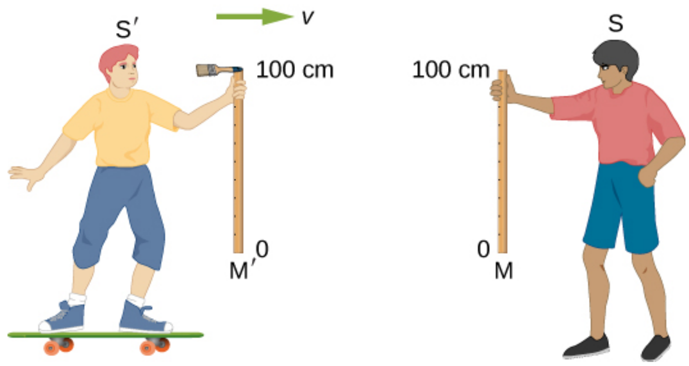


Figure 5.10 Meter sticks M and M' are stationary in the reference frames of observers S and S', respectively. As the sticks pass, a small brush attached to the 100-cm mark of M' paints a line on M.

Now when the boys approach each other, S', like S, sees a meter stick moving toward him with speed *v*. Because their situations are symmetric, each boy must make the same measurement of the stick in the other frame. So, if S measures stick M' to be less than 1.0 m long, S' must measure stick M to be also less than 1.0 m long, and S' must see his paintbrush pass over the top of stick M and not paint a line on it. In other words, after the same event, one boy sees a painted line on a stick, while the other does not see such a line on that same stick!

Einstein's first postulate requires that the laws of physics (as, for example, applied to painting) predict that S and S', who are both in inertial frames, make the same observations; that is, S and S' must either both see a line painted on stick M, or both not see that line. We are therefore forced to conclude our original assumption that S saw a line painted below the top of his stick was wrong! Instead, S finds the line painted right at the 100-cm mark on M. Then both boys will agree that a line is painted on M, and they will also agree that both sticks are exactly 1 m long. We conclude then that measurements of a transverse *length must be the same in different inertial frames*.

Example 5.5

Calculating Length Contraction

Suppose an astronaut, such as the twin in the twin paradox discussion, travels so fast that $\gamma = 30.00$. (a) The astronaut travels from Earth to the nearest star system, Alpha Centauri, 4.300 light years (ly) away as measured by an earthbound observer. How far apart are Earth and Alpha Centauri as measured by the astronaut? (b) In terms of *c*, what is the astronaut's velocity relative to Earth? You may neglect the motion of Earth relative to the sun (**Figure 5.11**).

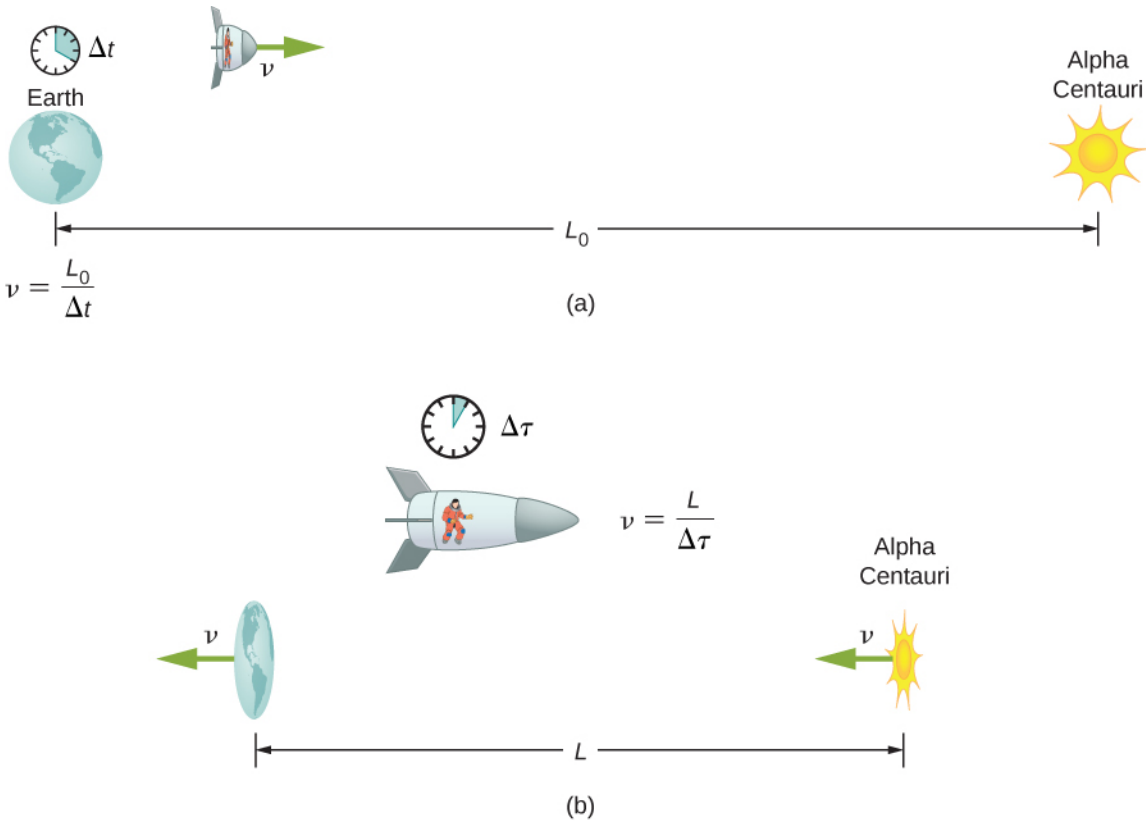


Figure 5.11 (a) The earthbound observer measures the proper distance between Earth and Alpha Centauri. (b) The astronaut observes a length contraction because Earth and Alpha Centauri move relative to her ship. She can travel this shorter distance in a smaller time (her proper time) without exceeding the speed of light.

Strategy

First, note that a light year (ly) is a convenient unit of distance on an astronomical scale—it is the distance light travels in a year. For part (a), the 4.300-ly distance between Alpha Centauri and Earth is the proper distance L_0, because it is measured by an earthbound observer to whom both stars are (approximately) stationary. To the astronaut, Earth and Alpha Centauri are moving past at the same velocity, so the distance between them is the contracted length *L*. In part (b), we are given γ, so we can find *v* by rearranging the definition of γ to express *v* in terms of *c*.

Solution for (a)

For part (a):

a. Identify the knowns: $L_0 = 4.300\text{ ly}; \gamma = 30.00.$

b. Identify the unknown: *L*.

c. Express the answer as an equation: $L = \frac{L_0}{\gamma}.$

d. Do the calculation:

$$\begin{aligned} L &= \frac{L_0}{\gamma} \\ &= \frac{4.300\text{ ly}}{30.00} \\ &= 0.1433\text{ ly.} \end{aligned}$$

Solution for (b)

For part (b):

a. Identify the known: $\gamma = 30.00.$

b. Identify the unknown: v in terms of c.

c. Express the answer as an equation. Start with:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Then solve for the unknown v/c by first squaring both sides and then rearranging:

$$\begin{aligned} \gamma^2 &= \frac{1}{1 - \frac{v^2}{c^2}} \\ \frac{v^2}{c^2} &= 1 - \frac{1}{\gamma^2} \\ \frac{v}{c} &= \sqrt{1 - \frac{1}{\gamma^2}}. \end{aligned}$$

d. Do the calculation:

$$\begin{aligned} \frac{v}{c} &= \sqrt{1 - \frac{1}{\gamma^2}} \\ &= \sqrt{1 - \frac{1}{(30.00)^2}} \\ &= 0.99944 \end{aligned}$$

or

$$v = 0.9994\,c.$$

Significance

Remember not to round off calculations until the final answer, or you could get erroneous results. This is especially true for special relativity calculations, where the differences might only be revealed after several decimal places. The relativistic effect is large here $(\gamma = 30.00)$, and we see that v is approaching (not equaling) the speed of light. Because the distance as measured by the astronaut is so much smaller, the astronaut can travel it in much less time in her frame.

People traveling at extremely high velocities could cover very large distances (thousands or even millions of light years) and age only a few years on the way. However, like emigrants in past centuries who left their home, these people would leave the Earth they know forever. Even if they returned, thousands to millions of years would have passed on Earth, obliterating most of what now exists. There is also a more serious practical obstacle to traveling at such velocities; immensely greater energies would be needed to achieve such high velocities than classical physics predicts can be attained. This will be discussed later in the chapter.

Why don't we notice length contraction in everyday life? The distance to the grocery store does not seem to depend on whether we are moving or not. Examining the equation $L = L_0\sqrt{1 - \frac{v^2}{c^2}}$, we see that at low velocities $(v \ll c)$, the lengths are nearly equal, which is the classical expectation. But length contraction is real, if not commonly experienced. For example, a charged particle such as an electron traveling at relativistic velocity has electric field lines that are compressed along the direction of motion as seen by a stationary observer (**Figure 5.12**). As the electron passes a detector, such as a coil of wire, its field interacts much more briefly, an effect observed at particle accelerators such as the 3-km-long Stanford Linear Accelerator (SLAC). In fact, to an electron traveling down the beam pipe at SLAC, the accelerator and Earth are all moving by and are length contracted. The relativistic effect is so great that the accelerator is only 0.5 m long to the electron. It is actually easier to get the electron beam down the pipe, because the beam does not have to be as precisely aimed to get down a short pipe as it would to get down a pipe 3 km long. This, again, is an experimental verification of the special theory of relativity.

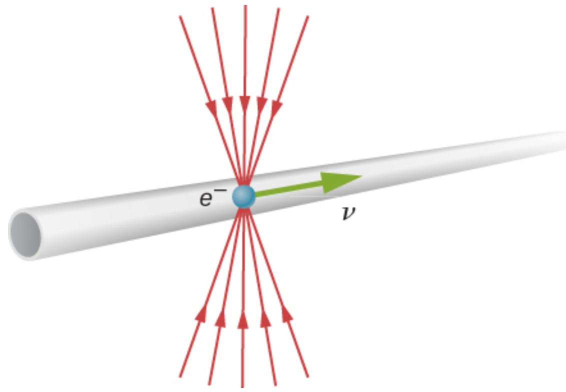


Figure 5.12 The electric field lines of a high-velocity charged particle are compressed along the direction of motion by length contraction, producing an observably different signal as the particle goes through a coil.

5.4 **Check Your Understanding** A particle is traveling through Earth's atmosphere at a speed of $0.750c$. To an earthbound observer, the distance it travels is 2.50 km. How far does the particle travel as viewed from the particle's reference frame?

5.5 | The Lorentz Transformation

Learning Objectives

- Describe the Galilean transformation of classical mechanics, relating the position, time, velocities, and accelerations measured in different inertial frames
- Derive the corresponding Lorentz transformation equations, which, in contrast to the Galilean transformation, are consistent with special relativity
- Explain the Lorentz transformation and many of the features of relativity in terms of four-dimensional space-time

We have used the postulates of relativity to examine, in particular examples, how observers in different frames of reference measure different values for lengths and the time intervals. We can gain further insight into how the postulates of relativity change the Newtonian view of time and space by examining the transformation equations that give the space and time coordinates of events in one inertial reference frame in terms of those in another. We first examine how position and time coordinates transform between inertial frames according to the view in Newtonian physics. Then we examine how this has to be changed to agree with the postulates of relativity. Finally, we examine the resulting Lorentz transformation equations and some of their consequences in terms of four-dimensional space-time diagrams, to support the view that the consequences of special relativity result from the properties of time and space itself, rather than electromagnetism.

The Galilean Transformation Equations

An **event** is specified by its location and time (x, y, z, t) relative to one particular inertial frame of reference S. As an example, (x, y, z, t) could denote the position of a particle at time t, and we could be looking at these positions for many different times to follow the motion of the particle. Suppose a second frame of reference S' moves with velocity v with respect to the first. For simplicity, assume this relative velocity is along the x-axis. The relation between the time and coordinates in the two frames of reference is then

$$x = x' + vt, \quad y = y', \quad z = z'.$$

Implicit in these equations is the assumption that time measurements made by observers in both S and S' are the same. That is,

$$t = t'.$$

These four equations are known collectively as the **Galilean transformation**.

We can obtain the Galilean velocity and acceleration transformation equations by differentiating these equations with respect to time. We use *u* for the velocity of a particle throughout this chapter to distinguish it from *v*, the relative velocity of two reference frames. Note that, for the Galilean transformation, the increment of time used in differentiating to calculate the particle velocity is the same in both frames, $dt = dt'$. Differentiation yields

$$u_x = u'_x + v, \quad u_y = u'_y, \quad u_z = u'_z$$

and

$$a_x = a'_x, \quad a_y = a'_y, \quad a_z = a'_z.$$

We denote the velocity of the particle by *u* rather than *v* to avoid confusion with the velocity *v* of one frame of reference with respect to the other. Velocities in each frame differ by the velocity that one frame has as seen from the other frame. Observers in both frames of reference measure the same value of the acceleration. Because the mass is unchanged by the transformation, and distances between points are uncharged, observers in both frames see the same forces $F = ma$ acting between objects and the same form of Newton's second and third laws in all inertial frames. The laws of mechanics are consistent with the first postulate of relativity.

The Lorentz Transformation Equations

The Galilean transformation nevertheless violates Einstein's postulates, because the velocity equations state that a pulse of light moving with speed *c* along the *x*-axis would travel at speed $c - v$ in the other inertial frame. Specifically, the spherical pulse has radius $r = ct$ at time *t* in the unprimed frame, and also has radius $r' = ct'$ at time t' in the primed frame. Expressing these relations in Cartesian coordinates gives

$$\begin{aligned} x^2 + y^2 + z^2 - c^2t^2 &= 0 \\ x'^2 + y'^2 + z'^2 - c^2t'^2 &= 0. \end{aligned}$$

The left-hand sides of the two expressions can be set equal because both are zero. Because $y = y'$ and $z = z'$, we obtain

$$x^2 - c^2t^2 = x'^2 - c^2t'^2. \quad (5.5)$$

This cannot be satisfied for nonzero relative velocity *v* of the two frames if we assume the Galilean transformation results in $t = t'$ with $x = x' + vt'$.

To find the correct set of transformation equations, assume the two coordinate systems *S* and S' in **Figure 5.13**. First suppose that an event occurs at $(x', 0, 0, t')$ in S' and at $(x, 0, 0, t)$ in *S*, as depicted in the figure.

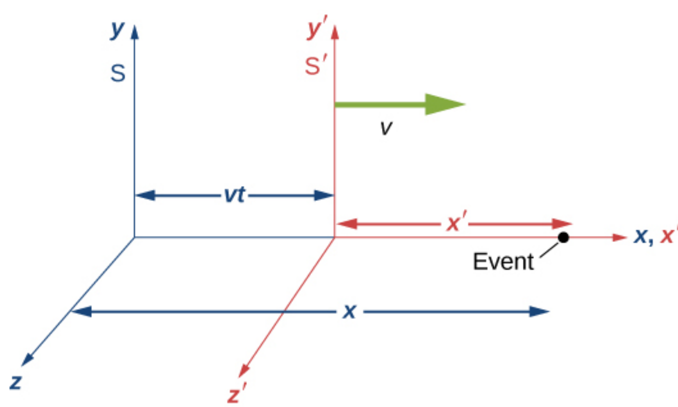


Figure 5.13 An event occurs at (*x*, 0, 0, *t*) in *S* and at $(x', 0, 0, t')$ in S'. The Lorentz transformation equations relate events in the two systems.

Suppose that at the instant that the origins of the coordinate systems in *S* and S' coincide, a flash bulb emits a spherically spreading pulse of light starting from the origin. At time *t*, an observer in *S* finds the origin of S' to be at $x = vt$. With the help of a friend in *S*, the S' observer also measures the distance from the event to the origin of S' and finds it to be

$x'\sqrt{1 - v^2/c^2}$. This follows because we have already shown the postulates of relativity to imply length contraction. Thus the position of the event in S is

$$x = vt + x'\sqrt{1 - v^2/c^2}$$

and

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}.$$

The postulates of relativity imply that the equation relating distance and time of the spherical wave front:

$$x^2 + y^2 + z^2 - c^2t^2 = 0$$

must apply both in terms of primed and unprimed coordinates, which was shown above to lead to **Equation 5.5**:

$$x^2 - c^2t^2 = x'^2 - c^2t'^2.$$

We combine this with the equation relating x and x' to obtain the relation between t and t':

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}.$$

The equations relating the time and position of the events as seen in S are then

$$\begin{aligned} t &= \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \\ x &= \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \\ y &= y' \\ z &= z'. \end{aligned}$$

This set of equations, relating the position and time in the two inertial frames, is known as the **Lorentz transformation**. They are named in honor of H.A. Lorentz (1853–1928), who first proposed them. Interestingly, he justified the transformation on what was eventually discovered to be a fallacious hypothesis. The correct theoretical basis is Einstein's special theory of relativity.

The reverse transformation expresses the variables in S in terms of those in S'. Simply interchanging the primed and unprimed variables and substituting gives:

$$\begin{aligned} t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \\ x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z. \end{aligned}$$

Example 5.6

Using the Lorentz Transformation for Time

Spacecraft S' is on its way to Alpha Centauri when Spacecraft S passes it at relative speed $c/2$. The captain of S' sends a radio signal that lasts 1.2 s according to that ship's clock. Use the Lorentz transformation to find the time interval of the signal measured by the communications officer of spaceship S.

Solution

a. Identify the known: $\Delta t' = t_2{}' - t_1{}' = 1.2\text{ s}$; $\Delta x' = x'_2 - x'_1 = 0$.

b. Identify the unknown: $\Delta t = t_2 - t_1$.

c. Express the answer as an equation. The time signal starts as (x', t_1') and stops at (x', t_2'). Note that the x' coordinate of both events is the same because the clock is at rest in S'. Write the first Lorentz transformation equation in terms of $\Delta t = t_2 - t_1$, $\Delta x = x_2 - x_1$, and similarly for the primed coordinates, as:

$$\Delta t = \frac{\Delta t' + v\Delta x'/c^2}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Because the position of the clock in S' is fixed, $\Delta x' = 0$, and the time interval Δt becomes:

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

d. Do the calculation.
With $\Delta t' = 1.2\text{ s}$ this gives:

$$\Delta t = \frac{1.2\text{ s}}{\sqrt{1 - \left(\frac{1}{2}\right)^2}} = 1.6\text{ s}.$$

Note that the Lorentz transformation reproduces the time dilation equation.

Example 5.7

Using the Lorentz Transformation for Length

A surveyor measures a street to be $L = 100\text{ m}$ long in Earth frame S. Use the Lorentz transformation to obtain an expression for its length measured from a spaceship S', moving by at speed 0.20*c*, assuming the *x* coordinates of the two frames coincide at time $t = 0$.

Solution

a. Identify the known: $L = 100\text{ m}$; $v = 0.20c$; $\Delta\tau = 0$.

b. Identify the unknown: L'.

c. Express the answer as an equation. The surveyor in frame S has measured the two ends of the stick simultaneously, and found them at rest at x_2 and x_1 a distance $L = x_2 - x_1 = 100\text{ m}$ apart. The spaceship crew measures the simultaneous location of the ends of the sticks in their frame. To relate the lengths recorded by observers in S' and S, respectively, write the second of the four Lorentz transformation equations as:

$$\begin{aligned} x'_2 - x'_1 &= \frac{x_2 - vt}{\sqrt{1 - v^2/c^2}} - \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}} \\ &= \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}} = \frac{L}{\sqrt{1 - v^2/c^2}}. \end{aligned}$$

d. Do the calculation. Because $x'_2 - x'_1 = 100\text{ m}$, the length of the moving stick is equal to:

$$\begin{aligned} L' &= (100\text{ m})\sqrt{1 - v^2/c^2} \\ &= (100\text{ m})\sqrt{1 - (0.20)^2} \\ &= 98.0\text{ m}. \end{aligned}$$

Note that the Lorentz transformation gave the length contraction equation for the street.

Example 5.8

Lorentz Transformation and Simultaneity

The observer shown in **Figure 5.14** standing by the railroad tracks sees the two bulbs flash simultaneously at both ends of the 26 m long passenger car when the middle of the car passes him at a speed of $c/2$. Find the separation in time between when the bulbs flashed as seen by the train passenger seated in the middle of the car.

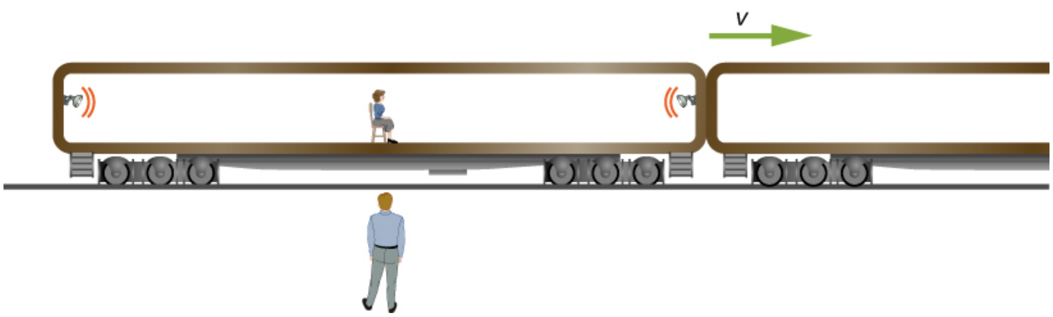


Figure 5.14 An person watching a train go by observes two bulbs flash simultaneously at opposite ends of a passenger car. There is another passenger inside of the car observing the same flashes but from a different perspective.

Solution

a. Identify the known: $\Delta t = 0$.

   Note that the spatial separation of the two events is between the two lamps, not the distance of the lamp to the passenger.

b. Identify the unknown: $\Delta t' = t'_2 - t'_1$.

   Again, note that the time interval is between the flashes of the lamps, not between arrival times for reaching the passenger.

c. Express the answer as an equation:

$$\Delta t = \frac{\Delta t' + v\Delta x'/c^2}{\sqrt{1 - v^2/c^2}}.$$

d. Do the calculation:

$$\begin{aligned} 0 &= \frac{\Delta t' + \frac{c}{2}(26\text{ m})/c^2}{\sqrt{1 - v^2/c^2}} \\ \Delta t' &= -\frac{26\text{ m/s}}{2c} = -\frac{26\text{ m/s}}{2\left(3.00\times10^8\text{ m/s}\right)} \\ \Delta t' &= -4.33\times10^{-8}\text{ s}. \end{aligned}$$

Significance

The sign indicates that the event with the larger x_2', namely, the flash from the right, is seen to occur first in the S' frame, as found earlier for this example, so that $t_2 < t_1$.

Space-time

Relativistic phenomena can be analyzed in terms of events in a four-dimensional space-time. When phenomena such as the twin paradox, time dilation, length contraction, and the dependence of simultaneity on relative motion are viewed in this way, they are seen to be characteristic of the nature of space and time, rather than specific aspects of electromagnetism.

In three-dimensional space, positions are specified by three coordinates on a set of Cartesian axes, and the displacement of one point from another is given by:

$$(\Delta x,\ \Delta y,\ \Delta z) = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1).$$

The distance Δr between the points is

$$\Delta r^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2.$$

The distance Δr is invariant under a rotation of axes. If a new set of Cartesian axes rotated around the origin relative to the original axes are used, each point in space will have new coordinates in terms of the new axes, but the distance $\Delta r'$ given by

$$\Delta r'^2 = (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2.$$

That has the same value that Δr^2 had. Something similar happens with the Lorentz transformation in space-time.

Define the separation between two events, each given by a set of *x*, *y*, *z*¸ and *ct* along a four-dimensional Cartesian system of axes in space-time, as

$$(\Delta x,\ \Delta y,\ \Delta z,\ c\Delta t) = \left(x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1,\ c(t_2 - t_1)\right).$$

Also define the space-time interval Δs between the two events as

$$\Delta s^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2.$$

If the two events have the same value of *ct* in the frame of reference considered, Δs would correspond to the distance Δr between points in space.

The path of a particle through space-time consists of the events (*x*, *y*, *z*¸ *ct*) specifying a location at each time of its motion. The path through space-time is called the **world line** of the particle. The world line of a particle that remains at rest at the same location is a straight line that is parallel to the time axis. If the particle moves at constant velocity parallel to the *x*-axis, its world line would be a sloped line $x = vt,$ corresponding to a simple displacement vs. time graph. If the particle accelerates, its world line is curved. The increment of *s* along the world line of the particle is given in differential form as

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2 - c^2(dt)^2.$$

Just as the distance Δr is invariant under rotation of the space axes, the space-time interval:

$$\Delta s^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2.$$

is invariant under the Lorentz transformation. This follows from the postulates of relativity, and can be seen also by substitution of the previous Lorentz transformation equations into the expression for the space-time interval:

$$\begin{aligned}\Delta s^2 &= (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2 \\ &= \left(\frac{\Delta x' + v\Delta t'}{\sqrt{1 - v^2/c^2}}\right)^2 + (\Delta y')^2 + (\Delta z')^2 - \left(c\frac{\Delta t' + \frac{v\Delta x'}{c^2}}{\sqrt{1 - v^2/c^2}}\right)^2 \\ &= (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 - (c\Delta t')^2 \\ &= \Delta s'^2.\end{aligned}$$

In addition, the Lorentz transformation changes the coordinates of an event in time and space similarly to how a three-dimensional rotation changes old coordinates into new coordinates:

Lorentz transformation	**Axis – rotation around *z*-axis**
(x, t coordinates):	(x, y coordinates):
$x' = (\gamma)x + (-\beta\gamma)ct$	$x' = (\cos\theta)x + (\sin\theta)y$
$ct' = (-\beta\gamma)x + (\gamma)ct$	$y' = (-\sin\theta)x + (\cos\theta)y$

where $\gamma = \dfrac{1}{\sqrt{1 - \beta^2}};\quad \beta = v/c.$

Lorentz transformations can be regarded as generalizations of spatial rotations to space-time. However, there are some differences between a three-dimensional axis rotation and a Lorentz transformation involving the time axis, because of differences in how the metric, or rule for measuring the displacements Δr and Δs, differ. Although Δr is invariant under spatial rotations and Δs is invariant also under Lorentz transformation, the Lorentz transformation involving the time axis does not preserve some features, such as the axes remaining perpendicular or the length scale along each axis remaining the same.

Note that the quantity Δs^2 can have either sign, depending on the coordinates of the space-time events involved. For pairs of events that give it a negative sign, it is useful to define $\Delta\tau^2$ as $-\Delta s^2$. The significance of $\Delta\tau$ as just defined follows by noting that in a frame of reference where the two events occur at the same location, we have $\Delta x = \Delta y = \Delta z = 0$ and therefore (from the equation for $\Delta s^2 = -\Delta\tau^2$):

$$\Delta\tau^2 = -\Delta s^2 = (\Delta t)^2.$$

Therefore $\Delta\tau$ is the time interval Δt in the frame of reference where both events occur at the same location. It is the same interval of proper time discussed earlier. It also follows from the relation between Δs and that $\Delta\tau$ that because Δs is Lorentz invariant, the proper time is also Lorentz invariant. All observers in all inertial frames agree on the proper time intervals between the same two events.

5.5 Check Your Understanding Show that if a time increment dt elapses for an observer who sees the particle moving with velocity v, it corresponds to a proper time particle increment for the particle of $d\tau = \gamma dt$.

The light cone

We can deal with the difficulty of visualizing and sketching graphs in four dimensions by imagining the three spatial coordinates to be represented collectively by a horizontal axis, and the vertical axis to be the *ct*-axis. Starting with a particular event in space-time as the origin of the space-time graph shown, the world line of a particle that remains at rest at the initial location of the event at the origin then is the time axis. Any plane through the time axis parallel to the spatial axes contains all the events that are simultaneous with each other and with the intersection of the plane and the time axis, as seen in the rest frame of the event at the origin.

It is useful to picture a light cone on the graph, formed by the world lines of all light beams passing through the origin event *A*, as shown in **Figure 5.15**. The light cone, according to the postulates of relativity, has sides at an angle of $45°$ if the time axis is measured in units of *ct*, and, according to the postulates of relativity, the light cone remains the same in all inertial frames. Because the event *A* is arbitrary, every point in the space-time diagram has a light cone associated with it.

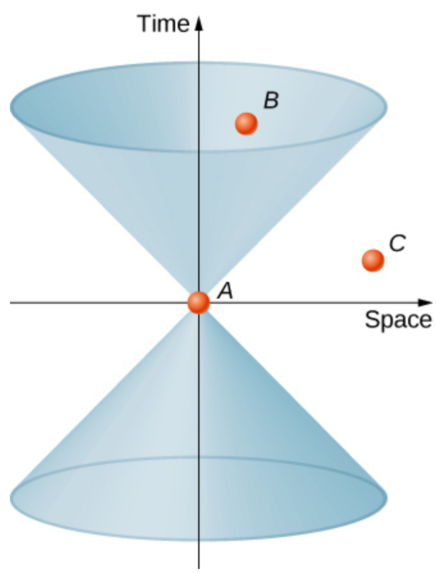


Figure 5.15 The light cone consists of all the world lines followed by light from the event *A* at the vertex of the cone.

Consider now the world line of a particle through space-time. Any world line outside of the cone, such as one passing from *A* through *C*, would involve speeds greater than *c*, and would therefore not be possible. Events such as *C* that lie outside the light cone are said to have a space-like separation from event *A*. They are characterized by:

$$\Delta s_{AC}^2 = (x_A - x_B)^2 + (x_A - x_B)^2 + (x_A - x_B)^2 - (c\Delta t)^2 > 0.$$

An event like *B* that lies in the upper cone is reachable without exceeding the speed of light in vacuum, and is characterized by

$$\Delta s_{AB}^2 = (x_A - x_B)^2 + (x_A - x_B)^2 + (x_A - x_B)^2 - (c\Delta t)^2 < 0.$$

The event is said to have a time-like separation from *A*. Time-like events that fall into the upper half of the light cone occur at greater values of *t* than the time of the event *A* at the vertex and are in the future relative to *A*. Events that have time-like separation from A and fall in the lower half of the light cone are in the past, and can affect the event at the origin. The region outside the light cone is labeled as neither past nor future, but rather as "elsewhere."

For any event that has a space-like separation from the event at the origin, it is possible to choose a time axis that will make the two events occur at the same time, so that the two events are simultaneous in some frame of reference. Therefore, which of the events with space-like separation comes before the other in time also depends on the frame of reference of the observer. Since space-like separations can be traversed only by exceeding the speed of light; this violation of which event can cause the other provides another argument for why particles cannot travel faster than the speed of light, as well as potential material for science fiction about time travel. Similarly for any event with time-like separation from the event at the origin, a frame of reference can be found that will make the events occur at the same location. Because the relations

$$\Delta s_{AC}^2 = (x_A - x_B)^2 + (x_A - x_B)^2 + (x_A - x_B)^2 - (c\Delta t)^2 > 0$$

and

$$\Delta s_{AB}^2 = (x_A - x_B)^2 + (x_A - x_B)^2 + (x_A - x_B)^2 - (c\Delta t)^2 < 0.$$

are Lorentz invariant, whether two events are time-like and can be made to occur at the same place or space-like and can be made to occur at the same time is the same for all observers. All observers in different inertial frames of reference agree on whether two events have a time-like or space-like separation.

The twin paradox seen in space-time

The twin paradox discussed earlier involves an astronaut twin traveling at near light speed to a distant star system, and returning to Earth. Because of time dilation, the space twin is predicted to age much less than the earthbound twin. This seems paradoxical because we might have expected at first glance for the relative motion to be symmetrical and naively thought it possible to also argue that the earthbound twin should age less.

To analyze this in terms of a space-time diagram, assume that the origin of the axes used is fixed in Earth. The world line of the earthbound twin is then along the time axis.

The world line of the astronaut twin, who travels to the distant star and then returns, must deviate from a straight line path in order to allow a return trip. As seen in **Figure 5.16**, the circumstances of the two twins are not at all symmetrical. Their paths in space-time are of manifestly different length. Specifically, the world line of the earthbound twin has length $2c\Delta t$, which then gives the proper time that elapses for the earthbound twin as $2\Delta t$. The distance to the distant star system is $\Delta x = v\Delta t$. The proper time that elapses for the space twin is $2\Delta\tau$ where

$$c^2\Delta\tau^2 = -\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2.$$

This is considerably shorter than the proper time for the earthbound twin by the ratio

$$\begin{aligned}\frac{c\Delta\tau}{c\Delta t} &= \sqrt{\frac{(c\Delta t)^2 - (\Delta x)^2}{(c\Delta t)^2}} = \sqrt{\frac{(c\Delta t)^2 - (v\Delta t)^2}{(c\Delta t)^2}} \\ &= \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{\gamma}.\end{aligned}$$

consistent with the time dilation formula. The twin paradox is therefore seen to be no paradox at all. The situation of the two twins is not symmetrical in the space-time diagram. The only surprise is perhaps that the seemingly longer path on the space-time diagram corresponds to the smaller proper time interval, because of how $\Delta\tau$ and Δs depend on Δx and Δt.

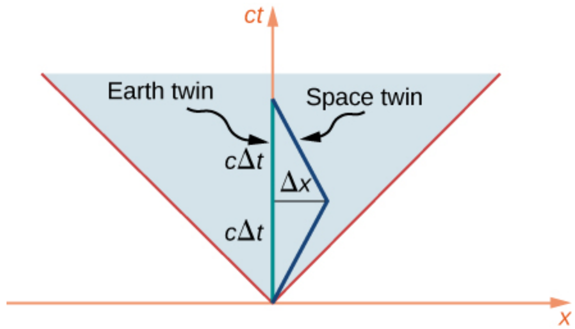


Figure 5.16 The space twin and the earthbound twin, in the twin paradox example, follow world lines of different length through space-time.

Lorentz transformations in space-time

We have already noted how the Lorentz transformation leaves

$$\Delta s^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2$$

unchanged and corresponds to a rotation of axes in the four-dimensional space-time. If the S and S' frames are in relative motion along their shared *x*-direction the space and time axes of S' are rotated by an angle α as seen from S, in the way shown in shown in **Figure 5.17**, where:

$$\tan\alpha = \frac{v}{c} = \beta.$$

This differs from a rotation in the usual three-dimension sense, insofar as the two space-time axes rotate toward each other symmetrically in a scissors-like way, as shown. The rotation of the time and space axes are both through the same angle. The mesh of dashed lines parallel to the two axes show how coordinates of an event would be read along the primed axes. This would be done by following a line parallel to the x' and one parallel to the t' -axis, as shown by the dashed lines. The length scale of both axes are changed by:

$$ct' = ct\sqrt{\frac{1+\beta^2}{1-\beta^2}}; \qquad x' = x\sqrt{\frac{1+\beta^2}{1-\beta^2}}.$$

The line labeled "$v = c$" at $45°$ to the *x*-axis corresponds to the edge of the light cone, and is unaffected by the Lorentz transformation, in accordance with the second postulate of relativity. The "$v = c$" line, and the light cone it represents, are the same for both the *S* and S' frame of reference.

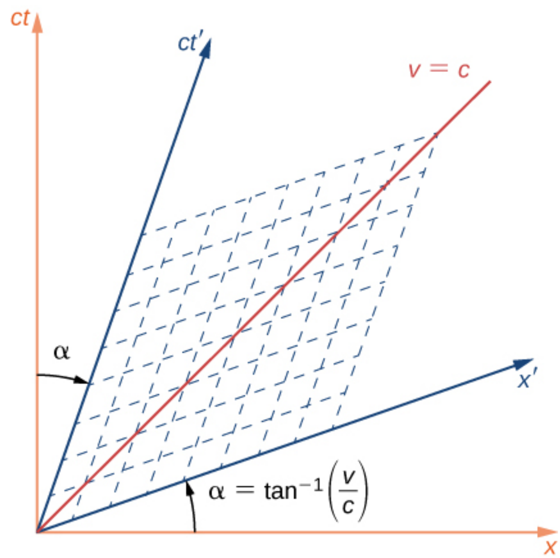


Figure 5.17 The Lorentz transformation results in new space and time axes rotated in a scissors-like way with respect to the original axes.

Simultaneity

Simultaneity of events at separated locations depends on the frame of reference used to describe them, as given by the scissors-like "rotation" to new time and space coordinates as described. If two events have the same *t* values in the unprimed frame of reference, they need not have the same values measured along the ct'-axis, and would then not be simultaneous in the primed frame.

As a specific example, consider the near-light-speed train in which flash lamps at the two ends of the car have flashed simultaneously in the frame of reference of an observer on the ground. The space-time graph is shown **Figure 5.18**. The flashes of the two lamps are represented by the dots labeled "Left flash lamp" and "Right flash lamp" that lie on the light cone in the past. The world line of both pulses travel along the edge of the light cone to arrive at the observer on the ground simultaneously. Their arrival is the event at the origin. They therefore had to be emitted simultaneously in the unprimed frame, as represented by the point labeled as *t*(both). But time is measured along the ct'-axis in the frame of reference of the observer seated in the middle of the train car. So in her frame of reference, the emission event of the bulbs labeled as t' (left) and t' (right) were not simultaneous.

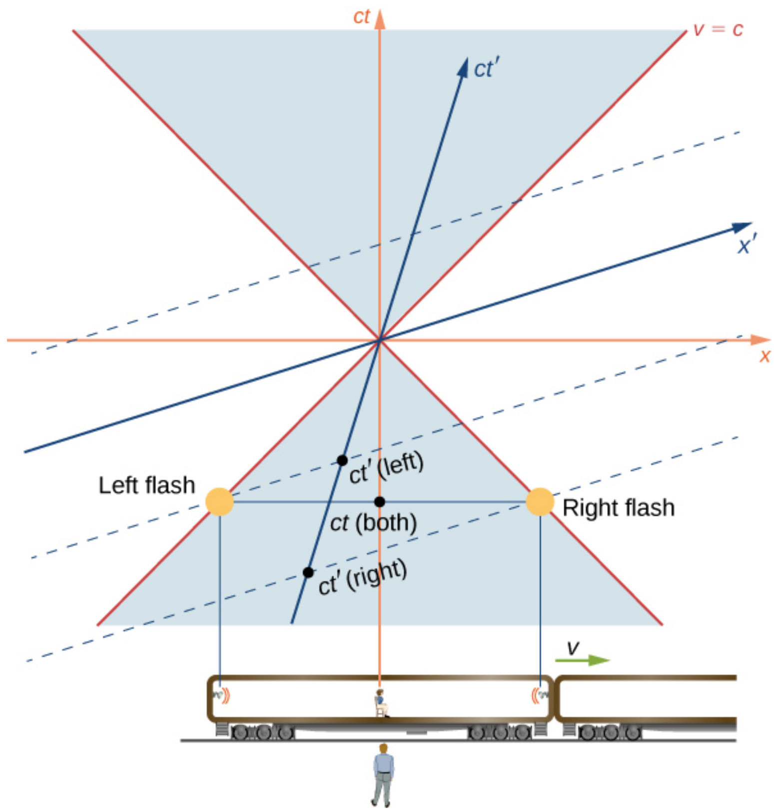


Figure 5.18 The train example revisited. The flashes occur at the same time t(both) along the time axis of the ground observer, but at different times, along the t' time axis of the passenger.

In terms of the space-time diagram, the two observers are merely using different time axes for the same events because they are in different inertial frames, and the conclusions of both observers are equally valid. As the analysis in terms of the space-time diagrams further suggests, the property of how simultaneity of events depends on the frame of reference results from the properties of space and time itself, rather than from anything specifically about electromagnetism.

5.6 | Relativistic Velocity Transformation

Learning Objectives

By the end of this section, you will be able to:

- Derive the equations consistent with special relativity for transforming velocities in one inertial frame of reference into another.
- Apply the velocity transformation equations to objects moving at relativistic speeds.
- Examine how the combined velocities predicted by the relativistic transformation equations compare with those expected classically.

Remaining in place in a kayak in a fast-moving river takes effort. The river current pulls the kayak along. Trying to paddle against the flow can move the kayak upstream relative to the water, but that only accounts for part of its velocity relative to the shore. The kayak's motion is an example of how velocities in Newtonian mechanics combine by vector addition. The kayak's velocity is the vector sum of its velocity relative to the water and the water's velocity relative to the riverbank. However, the relativistic addition of velocities is quite different.

Velocity Transformations

Imagine a car traveling at night along a straight road, as in **Figure 5.19**. The driver sees the light leaving the headlights at speed c within the car's frame of reference. If the Galilean transformation applied to light, then the light from the car's headlights would approach the pedestrian at a speed $u = v + c,$ contrary to Einstein's postulates.

Figure 5.19 According to experimental results and the second postulate of relativity, light from the car's headlights moves away from the car at speed c and toward the observer on the sidewalk at speed c.

Both the distance traveled and the time of travel are different in the two frames of reference, and they must differ in a way that makes the speed of light the same in all inertial frames. The correct rules for transforming velocities from one frame to another can be obtained from the Lorentz transformation equations.

Relativistic Transformation of Velocity

Suppose an object P is moving at constant velocity $\mathbf{u} = \left(u'_x, u'_y, u'_z\right)$ as measured in the S' frame. The S' frame is moving along its x'-axis at velocity v. In an increment of time dt', the particle is displaced by dx' along the x'-axis. Applying the Lorentz transformation equations gives the corresponding increments of time and displacement in the unprimed axes:

$$\begin{aligned} dt &= \gamma\left(dt' + v dx'/c^2\right) \\ dx &= \gamma(dx' + v dt') \\ dy &= dy' \\ dz &= dz'. \end{aligned}$$

The velocity components of the particle seen in the unprimed coordinate system are then

$$\begin{aligned} \frac{dx}{dt} &= \frac{\gamma(dx' + v dt')}{\gamma\left(dt' + v dx'/c^2\right)} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} \\ \frac{dy}{dt} &= \frac{dy'}{\gamma\left(dt' + v dx'/c^2\right)} = \frac{\frac{dy'}{dt'}}{\gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)} \\ \frac{dz}{dt} &= \frac{dz'}{\gamma\left(dt' + v dx'/c^2\right)} = \frac{\frac{dz'}{dt'}}{\gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)}. \end{aligned}$$

We thus obtain the equations for the velocity components of the object as seen in frame S:

$$u_x = \left(\frac{u'_x + v}{1 + vu'_x/c^2}\right), \quad u_y = \left(\frac{u'_y/\gamma}{1 + vu'_x/c^2}\right), \quad u_z = \left(\frac{u'_z/\gamma}{1 + vu'_x/c^2}\right).$$

Compare this with how the Galilean transformation of classical mechanics says the velocities transform, by adding simply as vectors:

$$u_x = u'_x + u, \quad u_y = u'_y, \quad u_z = u'_z.$$

When the relative velocity of the frames is much smaller than the speed of light, that is, when $v \ll c$, the special relativity velocity addition law reduces to the Galilean velocity law. When the speed v of S' relative to S is comparable to the speed of light, the **relativistic velocity addition** law gives a much smaller result than the **classical (Galilean) velocity addition** does.

Example 5.9

Velocity Transformation Equations for Light

Suppose a spaceship heading directly toward Earth at half the speed of light sends a signal to us on a laser-produced beam of light (Figure 5.20). Given that the light leaves the ship at speed c as observed from the ship, calculate the speed at which it approaches Earth.

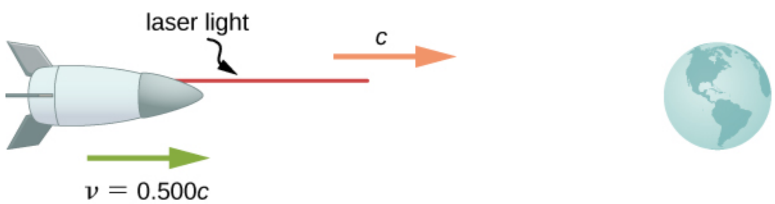


Figure 5.20 How fast does a light signal approach Earth if sent from a spaceship traveling at 0.500c?

Strategy

Because the light and the spaceship are moving at relativistic speeds, we cannot use simple velocity addition. Instead, we determine the speed at which the light approaches Earth using relativistic velocity addition.

Solution

a. Identify the knowns: $v = 0.500c$; $u' = c$.

b. Identify the unknown: u.

c. Express the answer as an equation: $u = \dfrac{v + u'}{1 + \frac{vu'}{c^2}}$.

d. Do the calculation:

$$\begin{aligned} u &= \frac{v + u'}{1 + \frac{vu'}{c^2}} \\ &= \frac{0.500c + c}{1 + \frac{(0.500c)(c)}{c^2}} \\ &= \frac{(0.500 + 1)c}{\left(\frac{c^2 + 0.500c^2}{c^2}\right)} \\ &= c. \end{aligned}$$

Significance

Relativistic velocity addition gives the correct result. Light leaves the ship at speed c and approaches Earth at speed c. The speed of light is independent of the relative motion of source and observer, whether the observer is on the ship or earthbound.

Velocities cannot add to greater than the speed of light, provided that v is less than c and u' does not exceed c. The following example illustrates that relativistic velocity addition is not as symmetric as classical velocity addition.

Example 5.10

Relativistic Package Delivery

Suppose the spaceship in the previous example approaches Earth at half the speed of light and shoots a canister at a speed of $0.750c$ (Figure 5.21). (a) At what velocity does an earthbound observer see the canister if it is shot directly toward Earth? (b) If it is shot directly away from Earth?

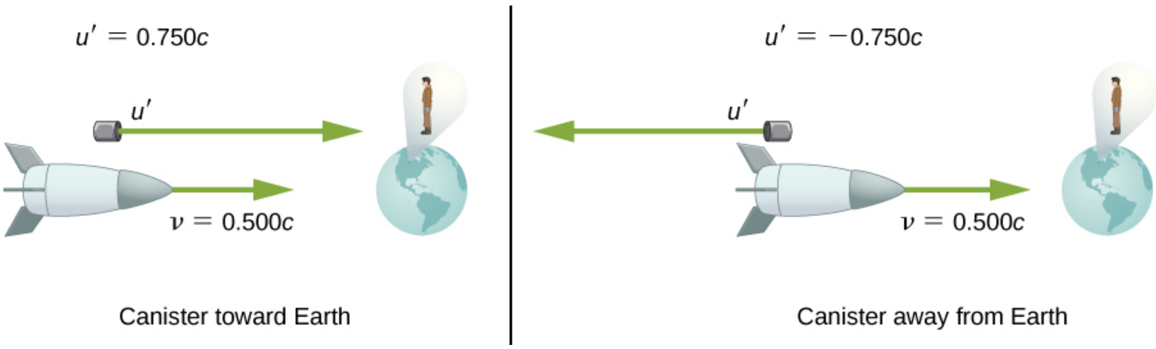


Figure 5.21 A canister is fired at $0.7500c$ toward Earth or away from Earth.

Strategy

Because the canister and the spaceship are moving at relativistic speeds, we must determine the speed of the canister by an earthbound observer using relativistic velocity addition instead of simple velocity addition.

Solution for (a)

a. Identify the knowns: $v = 0.500c$; $u' = 0.750c$.

b. Identify the unknown: u.

c. Express the answer as an equation: $u = \frac{v + u'}{1 + \frac{vu'}{c^2}}$.

d. Do the calculation:

$$\begin{aligned} u &= \frac{v + u'}{1 + \frac{vu'}{c^2}} \\ &= \frac{0.500c + 0.750c}{1 + \frac{(0.500c)(0.750c)}{c^2}} \\ &= 0.909c. \end{aligned}$$

Solution for (b)

a. Identify the knowns: $v = 0.500c$; $u' = -0.750c$.

b. Identify the unknown: u.

c. Express the answer as an equation: $u = \frac{v + u'}{1 + \frac{vu'}{c^2}}$.

d. Do the calculation:

$$\begin{aligned} u &= \frac{v + u'}{1 + \frac{vu'}{c^2}} \\ &= \frac{0.500c + (-0.750c)}{1 + \frac{(0.500c)(-0.750c)}{c^2}} \\ &= -0.400c. \end{aligned}$$

Significance

The minus sign indicates a velocity away from Earth (in the opposite direction from v), which means the canister is heading toward Earth in part (a) and away in part (b), as expected. But relativistic velocities do not add as simply as they do classically. In part (a), the canister does approach Earth faster, but at less than the vector sum of the velocities, which would give 1.250c. In part (b), the canister moves away from Earth at a velocity of $-0.400c$, which is *faster* than the −0.250c expected classically. The differences in velocities are not even symmetric: In part (a), an observer on Earth sees the canister and the ship moving apart at a speed of 0.409c, and at a speed of 0.900c in part (b).

5.6 Check Your Understanding Distances along a direction perpendicular to the relative motion of the two frames are the same in both frames. Why then are velocities perpendicular to the x-direction different in the two frames?

5.7 | Doppler Effect for Light

Learning Objectives

By the end of this section, you will be able to:

- Explain the origin of the shift in frequency and wavelength of the observed wavelength when observer and source moved toward or away from each other
- Derive an expression for the relativistic Doppler shift
- Apply the Doppler shift equations to real-world examples

As discussed in the chapter on sound, if a source of sound and a listener are moving farther apart, the listener encounters fewer cycles of a wave in each second, and therefore lower frequency, than if their separation remains constant. For the same reason, the listener detects a higher frequency if the source and listener are getting closer. The resulting Doppler shift in detected frequency occurs for any form of wave. For sound waves, however, the equations for the Doppler shift differ markedly depending on whether it is the source, the observer, or the air, which is moving. Light requires no medium, and the Doppler shift for light traveling in vacuum depends only on the relative speed of the observer and source.

The Relativistic Doppler Effect

Suppose an observer in S sees light from a source in S' moving away at velocity v (**Figure 5.22**). The wavelength of the light could be measured within S'—for example, by using a mirror to set up standing waves and measuring the distance between nodes. These distances are proper lengths with S' as their rest frame, and change by a factor $\sqrt{1 - v^2/c^2}$ when measured in the observer's frame S, where the ruler measuring the wavelength in S' is seen as moving.

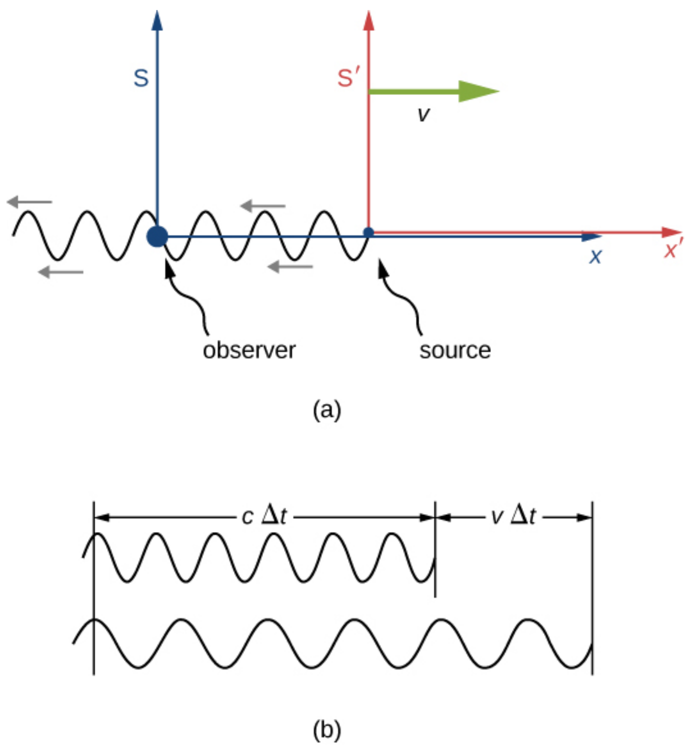


Figure 5.22 (a) When a light wave is emitted by a source fixed in the moving inertial frame S', the observer in *S* sees the wavelength measured in S'. to be shorter by a factor $\sqrt{1 - v^2/c^2}$. (b) Because the observer sees the source moving away within *S*, the wave pattern reaching the observer in *S* is also stretched by the factor $(c\Delta t + v\Delta t)/(c\Delta t) = 1 + v/c$.

If the source were stationary in *S*, the observer would see a length $c\Delta t$ of the wave pattern in time Δt. But because of the motion of S' relative to *S*, considered solely within *S*, the observer sees the wave pattern, and therefore the wavelength, stretched out by a factor of

$$\frac{c\Delta t_{\text{period}} + v\Delta t_{\text{period}}}{c\Delta t_{\text{period}}} = 1 + \frac{v}{c}$$

as illustrated in (b) of **Figure 5.22**. The overall increase from both effects gives

$$\lambda_{\text{obs}} = \lambda_{\text{src}}\left(1 + \frac{v}{c}\right)\sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} = \lambda_{\text{src}}\left(1 + \frac{v}{c}\right)\sqrt{\frac{1}{\left(1 + \frac{v}{c}\right)\left(1 - \frac{v}{c}\right)}} = \lambda_{\text{src}}\sqrt{\frac{\left(1 + \frac{v}{c}\right)}{\left(1 - \frac{v}{c}\right)}}$$

where λ_{src} is the wavelength of the light seen by the source in S' and λ_{obs} is the wavelength that the observer detects within *S*.

Red Shifts and Blue Shifts

The observed wavelength λ_{obs} of electromagnetic radiation is longer (called a “red shift”) than that emitted by the source when the source moves away from the observer. Similarly, the wavelength is shorter (called a “blue shift”) when the source moves toward the observer. The amount of change is determined by

$$\lambda_{\text{obs}} = \lambda_s\sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$

where λ_s is the wavelength in the frame of reference of the source, and v is the relative velocity of the two frames S and S'. The velocity v is positive for motion away from an observer and negative for motion toward an observer. In terms of source frequency and observed frequency, this equation can be written as

$$f_{obs} = f_s \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}.$$

Notice that the signs are different from those of the wavelength equation.

Example 5.11

Calculating a Doppler Shift

Suppose a galaxy is moving away from Earth at a speed $0.825c$. It emits radio waves with a wavelength of 0.525 m. What wavelength would we detect on Earth?

Strategy

Because the galaxy is moving at a relativistic speed, we must determine the Doppler shift of the radio waves using the relativistic Doppler shift instead of the classical Doppler shift.

Solution

a. Identify the knowns: $u = 0.825c$; $\lambda_s = 0.525$ m.

b. Identify the unknown: λ_{obs}.

c. Express the answer as an equation:

$$\lambda_{obs} = \lambda_s \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}.$$

d. Do the calculation:

$$\begin{aligned} \lambda_{obs} &= \lambda_s \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} \\ &= (0.525 \text{ m}) \sqrt{\frac{1 + \frac{0.825c}{c}}{1 - \frac{0.825c}{c}}} \\ &= 1.70 \text{ m}. \end{aligned}$$

Significance

Because the galaxy is moving away from Earth, we expect the wavelengths of radiation it emits to be redshifted. The wavelength we calculated is 1.70 m, which is redshifted from the original wavelength of 0.525 m. You will see in **Particle Physics and Cosmology** that detecting redshifted radiation led to present-day understanding of the origin and evolution of the universe.

5.7 Check Your Understanding Suppose a space probe moves away from Earth at a speed $0.350c$. It sends a radio-wave message back to Earth at a frequency of 1.50 GHz. At what frequency is the message received on Earth?

The relativistic Doppler effect has applications ranging from Doppler radar storm monitoring to providing information on the motion and distance of stars. We describe some of these applications in the exercises.

5.8 | Relativistic Momentum

Learning Objectives

By the end of this section, you will be able to:

- Define relativistic momentum in terms of mass and velocity
- Show how relativistic momentum relates to classical momentum
- Show how conservation of relativistic momentum limits objects with mass to speeds less than c

Momentum is a central concept in physics. The broadest form of Newton’s second law is stated in terms of momentum. Momentum is conserved whenever the net external force on a system is zero. This makes momentum conservation a fundamental tool for analyzing collisions (**Figure 5.23**). Much of what we know about subatomic structure comes from the analysis of collisions of accelerator-produced relativistic particles, and momentum conservation plays a crucial role in this analysis.

Figure 5.23 Momentum is an important concept for these football players from the University of California at Berkeley and the University of California at Davis. A player with the same velocity but greater mass collides with greater impact because his momentum is greater. For objects moving at relativistic speeds, the effect is even greater.

The first postulate of relativity states that the laws of physics are the same in all inertial frames. Does the law of conservation of momentum survive this requirement at high velocities? It can be shown that the momentum calculated as merely $\vec{\mathbf{p}} = m\frac{d\vec{\mathbf{x}}}{dt}$, even if it is conserved in one frame of reference, may not be conserved in another after applying the Lorentz transformation to the velocities. The correct equation for momentum can be shown, instead, to be the classical expression in terms of the increment $d\tau$ of proper time of the particle, observed in the particle’s rest frame:

$$\begin{aligned}\vec{\mathbf{p}} &= m\frac{d\vec{\mathbf{x}}}{d\tau} = m\frac{d\vec{\mathbf{x}}}{dt}\frac{dt}{d\tau} \\ &= m\frac{d\vec{\mathbf{x}}}{dt}\frac{1}{\sqrt{1-u^2/c^2}} \\ &= \frac{m\vec{\mathbf{u}}}{\sqrt{1-u^2/c^2}} = \gamma m\vec{\mathbf{u}}.\end{aligned}$$

Relativistic Momentum

Relativistic momentum $\vec{\mathbf{p}}$ is classical momentum multiplied by the relativistic factor γ:

$$\vec{\mathbf{p}} = \gamma m\vec{\mathbf{u}} \tag{5.6}$$

where m is the **rest mass** of the object, $\vec{\mathbf{u}}$ is its velocity relative to an observer, and γ is the relativistic factor:

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}. \tag{5.7}$$

Note that we use u for velocity here to distinguish it from relative velocity v between observers. The factor γ that occurs here has the same form as the previous relativistic factor γ except that it is now in terms of the velocity of the particle u instead of the relative velocity v of two frames of reference.

With p expressed in this way, total momentum p_{tot} is conserved whenever the net external force is zero, just as in classical physics. Again we see that the relativistic quantity becomes virtually the same as the classical quantity at low velocities, where u/c is small and γ is very nearly equal to 1. Relativistic momentum has the same intuitive role as classical momentum. It is greatest for large masses moving at high velocities, but because of the factor γ, relativistic momentum approaches infinity as u approaches c (**Figure 5.24**). This is another indication that an object with mass cannot reach the speed of light. If it did, its momentum would become infinite—an unreasonable value.

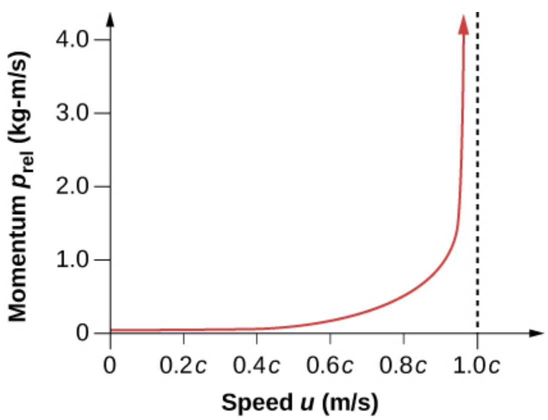


Figure 5.24 Relativistic momentum approaches infinity as the velocity of an object approaches the speed of light.

The relativistically correct definition of momentum as $p = \gamma mu$ is sometimes taken to imply that mass varies with velocity: $m_{\text{var}} = \gamma m$, particularly in older textbooks. However, note that m is the mass of the object as measured by a person at rest relative to the object. Thus, m is defined to be the rest mass, which could be measured at rest, perhaps using gravity. When a mass is moving relative to an observer, the only way that its mass can be determined is through collisions or other means involving momentum. Because the mass of a moving object cannot be determined independently of momentum, the only meaningful mass is rest mass. Therefore, when we use the term "mass," assume it to be identical to "rest mass."

Relativistic momentum is defined in such a way that conservation of momentum holds in all inertial frames. Whenever the net external force on a system is zero, relativistic momentum is conserved, just as is the case for classical momentum. This has been verified in numerous experiments.

5.8 Check Your Understanding What is the momentum of an electron traveling at a speed 0.985c? The rest mass of the electron is 9.11×10^{-31} kg.

5.9 | Relativistic Energy

Learning Objectives

By the end of this section, you will be able to:

- Explain how the work-energy theorem leads to an expression for the relativistic kinetic energy of an object
- Show how the relativistic energy relates to the classical kinetic energy, and sets a limit on the speed of any object with mass
- Describe how the total energy of a particle is related to its mass and velocity
- Explain how relativity relates to energy-mass equivalence, and some of the practical implications of energy-mass equivalence

The tokamak in **Figure 5.25** is a form of experimental fusion reactor, which can change mass to energy. Nuclear reactors are proof of the relationship between energy and matter.

Conservation of energy is one of the most important laws in physics. Not only does energy have many important forms, but each form can be converted to any other. We know that classically, the total amount of energy in a system remains constant. Relativistically, energy is still conserved, but energy-mass equivalence must now be taken into account, for example, in the reactions that occur within a nuclear reactor. Relativistic energy is intentionally defined so that it is conserved in all inertial frames, just as is the case for relativistic momentum. As a consequence, several fundamental quantities are related in ways not known in classical physics. All of these relationships have been verified by experimental results and have fundamental consequences. The altered definition of energy contains some of the most fundamental and spectacular new insights into nature in recent history.

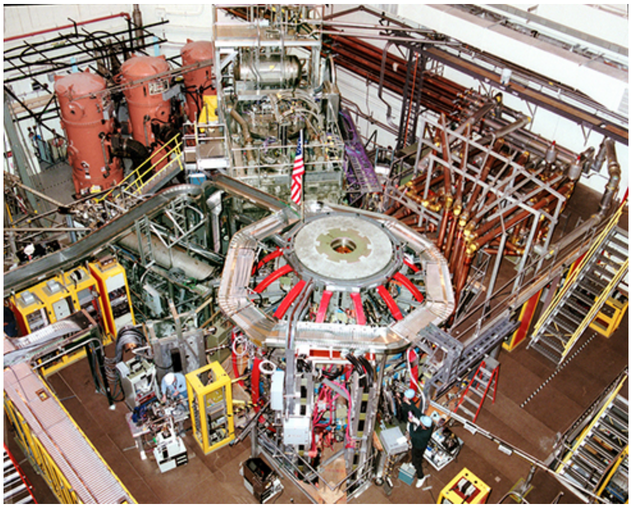


Figure 5.25 The National Spherical Torus Experiment (NSTX) is a fusion reactor in which hydrogen isotopes undergo fusion to produce helium. In this process, a relatively small mass of fuel is converted into a large amount of energy. (credit: Princeton Plasma Physics Laboratory)

Kinetic Energy and the Ultimate Speed Limit

The first postulate of relativity states that the laws of physics are the same in all inertial frames. Einstein showed that the law of conservation of energy of a particle is valid relativistically, but for energy expressed in terms of velocity and mass in a way consistent with relativity.

Consider first the relativistic expression for the kinetic energy. We again use u for velocity to distinguish it from relative velocity v between observers. Classically, kinetic energy is related to mass and speed by the familiar expression $K = \frac{1}{2}mu^2$. The corresponding relativistic expression for kinetic energy can be obtained from the work-energy theorem. This theorem states that the net work on a system goes into kinetic energy. Specifically, if a force, expressed as $\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} = m\frac{d(\gamma\vec{\mathbf{u}})}{dt}$, accelerates a particle from rest to its final velocity, the work done on the particle should be equal to its final kinetic energy. In mathematical form, for one-dimensional motion:

$$\begin{aligned} K &= \int F dx = \int m\frac{d}{dt}(\gamma u)dx \\ &= m\int \frac{d(\gamma u)}{dt}\frac{dx}{dt}dt = m\int u\frac{d}{dt}\left(\frac{u}{\sqrt{1-(u/c)^2}}\right)dt. \end{aligned}$$

Integrate this by parts to obtain

$$\begin{aligned} K &= \frac{mu^2}{\sqrt{1-(u/c)^2}}\Bigg|_{0u} - m\int \frac{u}{\sqrt{1-(u/c)^2}}\frac{du}{dt}dt \\ &= \frac{mu^2}{\sqrt{1-(u/c)^2}} - m\int \frac{u}{\sqrt{1-(u/c)^2}}du \\ &= \frac{mu^2}{\sqrt{1-(u/c)^2}} - mc^2\left(\sqrt{1-(u/c)^2}\right)\Bigg|_0^u \\ &= \frac{mu^2}{\sqrt{1-(u/c)^2}} + \frac{mc^2}{\sqrt{1-(u/c)^2}} - mc^2 \\ &= mc^2\left[\frac{(u^2/c^2) + 1 - (u^2/c^2)}{\sqrt{1-(u/c)^2}}\right] - mc^2 \\ K &= \frac{mc^2}{\sqrt{1-(u/c)^2}} - mc^2. \end{aligned}$$

Relativistic Kinetic Energy

Relativistic kinetic energy of any particle of mass m is

$$K_{\text{rel}} = (\gamma - 1)mc^2. \quad (5.8)$$

When an object is motionless, its speed is $u = 0$ and

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = 1$$

so that $K_{\text{rel}} = 0$ at rest, as expected. But the expression for relativistic kinetic energy (such as total energy and rest energy) does not look much like the classical $\frac{1}{2}mu^2$. To show that the expression for K_{rel} reduces to the classical expression for kinetic energy at low speeds, we use the binomial expansion to obtain an approximation for $(1+\varepsilon)^n$ valid for small ε:

$$(1+\varepsilon)^n = 1 + n\varepsilon + \frac{n(n-1)}{2!}\varepsilon^2 + \frac{n(n-1)(n-2)}{3!}\varepsilon^3 + \cdots \approx 1 + n\varepsilon$$

by neglecting the very small terms in ε^2 and higher powers of ε. Choosing $\varepsilon = -u^2/c^2$ and $n = -\frac{1}{2}$ leads to the conclusion that γ at nonrelativistic speeds, where $\varepsilon = u/c$ is small, satisfies

$$\gamma = \left(1 - u^2/c^2\right)^{-1/2} \approx 1 + \frac{1}{2}\left(\frac{u^2}{c^2}\right).$$

A binomial expansion is a way of expressing an algebraic quantity as a sum of an infinite series of terms. In some cases, as in the limit of small speed here, most terms are very small. Thus, the expression derived here for γ is not exact, but it is a very accurate approximation. Therefore, at low speed:

$$\gamma - 1 = \frac{1}{2}\left(\frac{u^2}{c^2}\right).$$

Entering this into the expression for relativistic kinetic energy gives

$$K_{\text{rel}} = \left[\frac{1}{2}\left(\frac{u^2}{c^2}\right)\right]mc^2 = \frac{1}{2}mu^2 = K_{\text{class}}.$$

That is, relativistic kinetic energy becomes the same as classical kinetic energy when $u << c$.

It is even more interesting to investigate what happens to kinetic energy when the speed of an object approaches the speed of light. We know that γ becomes infinite as *u* approaches *c*, so that K_{rel} also becomes infinite as the velocity approaches the speed of light (**Figure 5.26**). The increase in K_{rel} is far larger than in K_{class} as *v* approaches *c*. An infinite amount of work (and, hence, an infinite amount of energy input) is required to accelerate a mass to the speed of light.

The Speed of Light

No object with mass can attain the **speed of light**.

The speed of light is the ultimate speed limit for any particle having mass. All of this is consistent with the fact that velocities less than *c* always add to less than *c*. Both the relativistic form for kinetic energy and the ultimate speed limit being *c* have been confirmed in detail in numerous experiments. No matter how much energy is put into accelerating a mass, its velocity can only approach—not reach—the speed of light.

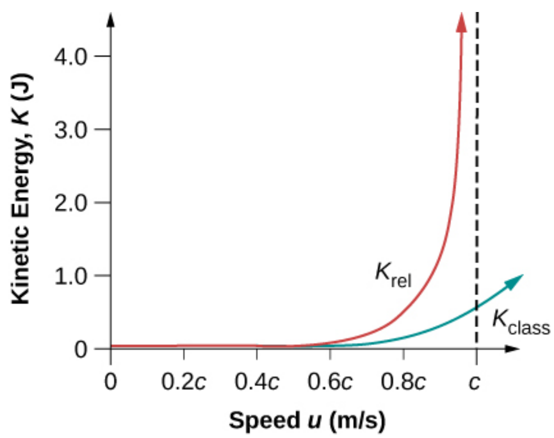


Figure 5.26 This graph of K_{rel} versus velocity shows how kinetic energy increases without bound as velocity approaches the speed of light. Also shown is K_{class}, the classical kinetic energy.

Example 5.12

Comparing Kinetic Energy

An electron has a velocity $v = 0.990c$. (a) Calculate the kinetic energy in MeV of the electron. (b) Compare this with the classical value for kinetic energy at this velocity. (The mass of an electron is 9.11×10^{-31} kg.)

Strategy

The expression for relativistic kinetic energy is always correct, but for (a), it must be used because the velocity is highly relativistic (close to *c*). First, we calculate the relativistic factor γ, and then use it to determine the relativistic kinetic energy. For (b), we calculate the classical kinetic energy (which would be close to the relativistic value if *v* were less than a few percent of *c*) and see that it is not the same.

Solution for (a)

For part (a):

a. Identify the knowns: $v = 0.990c$; $m = 9.11 \times 10^{-31}$ kg.

b. Identify the unknown: K_{rel}.

c. Express the answer as an equation: $K_{\text{rel}} = (\gamma - 1)mc^2$ with $\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$.

d. Do the calculation. First calculate γ. Keep extra digits because this is an intermediate calculation:

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \\ &= \frac{1}{\sqrt{1 - \frac{(0.990c)^2}{c^2}}} \\ &= 7.0888. \end{aligned}$$

Now use this value to calculate the kinetic energy:

$$\begin{aligned} K_{\text{rel}} &= (\gamma - 1)mc^2 \\ &= (7.0888 - 1)(9.11 \times 10^{-31}\ \text{kg})(3.00 \times 10^8\ \text{m/s}^2) \\ &= 4.9922 \times 10^{-13}\ \text{J}. \end{aligned}$$

e. Convert units:

$$\begin{aligned} K_{\text{rel}} &= (4.9922 \times 10^{-13}\ \text{J})\left(\frac{1\ \text{MeV}}{1.60 \times 10^{-13}\ \text{J}}\right) \\ &= 3.12\ \text{MeV}. \end{aligned}$$

Solution for (b)

For part (b):

a. List the knowns: $v = 0.990c$; $m = 9.11 \times 10^{-31}$ kg.

b. List the unknown: K_{rel}.

c. Express the answer as an equation: $K_{\text{class}} = \frac{1}{2}mu^2$.

d. Do the calculation:

$$\begin{aligned} K_{\text{class}} &= \frac{1}{2}mu^2 \\ &= \frac{1}{2}(9.11\times10^{-31}\ \text{kg})(0.990)^2(3.00\times10^8\ \text{m/s})^2 \\ &= 4.0179\times10^{-14}\ \text{J.} \end{aligned}$$

e. Convert units:

$$\begin{aligned} K_{\text{class}} &= 4.0179\times10^{-14}\ \text{J}\left(\frac{1\ \text{MeV}}{1.60\times10^{-13}\ \text{J}}\right) \\ &= 0.251\ \text{Mev.} \end{aligned}$$

Significance

As might be expected, because the velocity is 99.0% of the speed of light, the classical kinetic energy differs significantly from the correct relativistic value. Note also that the classical value is much smaller than the relativistic value. In fact, $K_{\text{rel}}/K_{\text{class}} = 12.4$ in this case. This illustrates how difficult it is to get a mass moving close to the speed of light. Much more energy is needed than predicted classically. Ever-increasing amounts of energy are needed to get the velocity of a mass a little closer to that of light. An energy of 3 MeV is a very small amount for an electron, and it can be achieved with present-day particle accelerators. SLAC, for example, can accelerate electrons to over $50\times10^9\,\text{eV} = 50{,}000\,\text{MeV}$.

Is there any point in getting *v* a little closer to *c* than 99.0% or 99.9%? The answer is yes. We learn a great deal by doing this. The energy that goes into a high-velocity mass can be converted into any other form, including into entirely new particles. In the Large Hadron Collider in **Figure 5.27**, charged particles are accelerated before entering the ring-like structure. There, two beams of particles are accelerated to their final speed of about 99.7% the speed of light in opposite directions, and made to collide, producing totally new species of particles. Most of what we know about the substructure of matter and the collection of exotic short-lived particles in nature has been learned this way. Patterns in the characteristics of these previously unknown particles hint at a basic substructure for all matter. These particles and some of their characteristics will be discussed in a later chapter on particle physics.

Figure 5.27 The European Organization for Nuclear Research (called CERN after its French name) operates the largest particle accelerator in the world, straddling the border between France and Switzerland.

Total Relativistic Energy

The expression for kinetic energy can be rearranged to:

$$E = \frac{mu^2}{\sqrt{1 - u^2/c^2}} = K + mc^2.$$

Einstein argued in a separate article, also later published in 1905, that if the energy of a particle changes by ΔE, its mass changes by $\Delta m = \Delta E/c^2$. Abundant experimental evidence since then confirms that mc^2 corresponds to the energy that the particle of mass m has when at rest. For example, when a neutral pion of mass m at rest decays into two photons, the photons have zero mass but are observed to have total energy corresponding to mc^2 for the pion. Similarly, when a particle of mass m decays into two or more particles with smaller total mass, the observed kinetic energy imparted to the products of the decay corresponds to the decrease in mass. Thus, E is the total relativistic energy of the particle, and mc^2 is its rest energy.

Total Energy

Total energy E of a particle is

$$E = \gamma mc^2 \tag{5.9}$$

where m is mass, c is the speed of light, $\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$, and u is the velocity of the mass relative to an observer.

Rest Energy

Rest energy of an object is

$$E_0 = mc^2. \tag{5.10}$$

This is the correct form of Einstein's most famous equation, which for the first time showed that energy is related to the mass of an object at rest. For example, if energy is stored in the object, its rest mass increases. This also implies that mass can be destroyed to release energy. The implications of these first two equations regarding relativistic energy are so broad that they were not completely recognized for some years after Einstein published them in 1905, nor was the experimental proof that they are correct widely recognized at first. Einstein, it should be noted, did understand and describe the meanings and implications of his theory.

Example 5.13

Calculating Rest Energy

Calculate the rest energy of a 1.00-g mass.

Strategy

One gram is a small mass—less than one-half the mass of a penny. We can multiply this mass, in SI units, by the speed of light squared to find the equivalent rest energy.

Solution

a. Identify the knowns: $m = 1.00 \times 10^{-3}\,\text{kg}$; $c = 3.00 \times 10^{8}\,\text{m/s}$.

b. Identify the unknown: E_0.

c. Express the answer as an equation: $E_0 = mc^2$.

d. Do the calculation:

$$\begin{aligned} E_0 &= mc^2 = (1.00 \times 10^{-3}\ \text{kg})(3.00 \times 10^{8}\ \text{m/s})^2 \\ &= 9.00 \times 10^{13}\ \text{kg} \cdot \text{m}^2/\text{s}^2. \end{aligned}$$

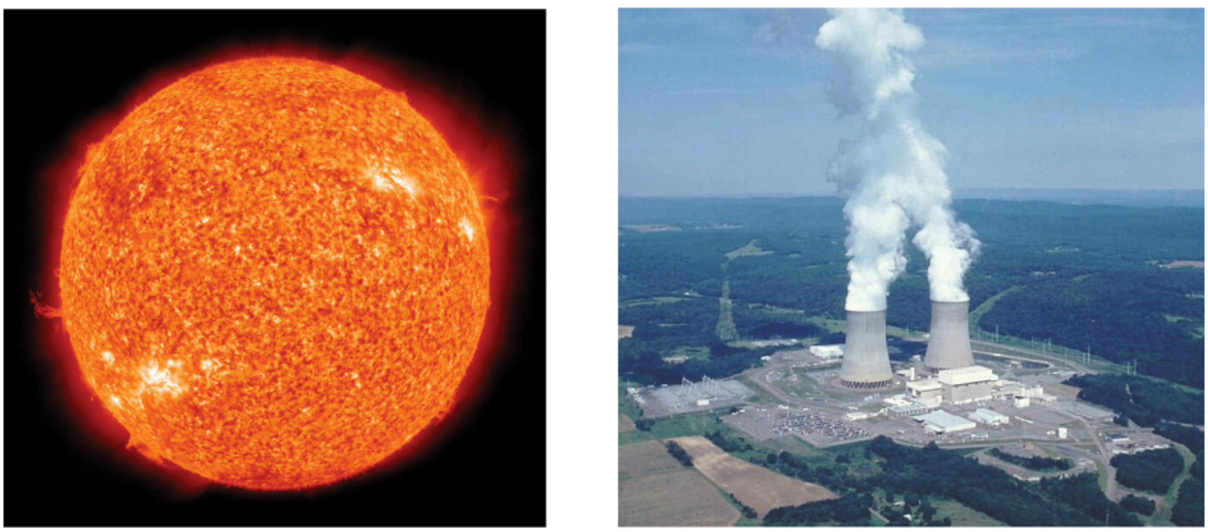
e. Convert units. Noting that $1\ \text{kg} \cdot \text{m}^2/\text{s}^2 = 1\ \text{J}$, we see the rest energy is:

$$E_0 = 9.00 \times 10^{13}\ \text{J}.$$

Significance

This is an enormous amount of energy for a 1.00-g mass. Rest energy is large because the speed of light c is a large number and c^2 is a very large number, so that mc^2 is huge for any macroscopic mass. The $9.00 \times 10^{13}\ \text{J}$ rest mass energy for 1.00 g is about twice the energy released by the Hiroshima atomic bomb and about 10,000 times the kinetic energy of a large aircraft carrier.

Today, the practical applications of *the conversion of mass into another form of energy*, such as in nuclear weapons and nuclear power plants, are well known. But examples also existed when Einstein first proposed the correct form of relativistic energy, and he did describe some of them. Nuclear radiation had been discovered in the previous decade, and it had been a mystery as to where its energy originated. The explanation was that, in some nuclear processes, a small amount of mass is destroyed and energy is released and carried by nuclear radiation. But the amount of mass destroyed is so small that it is difficult to detect that any is missing. Although Einstein proposed this as the source of energy in the radioactive salts then being studied, it was many years before there was broad recognition that mass could be and, in fact, commonly is, converted to energy (**Figure 5.28**).



(a) (b)

Figure 5.28 (a) The sun and (b) the Susquehanna Steam Electric Station both convert mass into energy—the sun via nuclear fusion, and the electric station via nuclear fission. (credit a: modification of work by NASA; credit b: modification of work by "ChNPP"/Wikimedia Commons)

Because of the relationship of rest energy to mass, we now consider mass to be a form of energy rather than something separate. There had not been even a hint of this prior to Einstein's work. Energy-mass equivalence is now known to be the source of the sun's energy, the energy of nuclear decay, and even one of the sources of energy keeping Earth's interior hot.

Stored Energy and Potential Energy

What happens to energy stored in an object at rest, such as the energy put into a battery by charging it, or the energy stored in a toy gun's compressed spring? The energy input becomes part of the total energy of the object and thus increases its rest mass. All stored and potential energy becomes mass in a system. In seeming contradiction, the principle of conservation of mass (meaning total mass is constant) was one of the great laws verified by nineteenth-century science. Why was it not noticed to be incorrect? The following example helps answer this question.

Example 5.14

Calculating Rest Mass

A car battery is rated to be able to move 600 ampere-hours $(\mathrm{A \cdot h})$ of charge at 12.0 V. (a) Calculate the increase in rest mass of such a battery when it is taken from being fully depleted to being fully charged, assuming none of the chemical reactants enter or leave the battery. (b) What percent increase is this, given that the battery's mass is 20.0 kg?

Strategy

In part (a), we first must find the energy stored as chemical energy E_{batt} in the battery, which equals the electrical energy the battery can provide. Because $E_{\text{batt}} = qV,$ we have to calculate the charge q in $600\ \mathrm{A \cdot h},$ which is the product of the current I and the time t. We then multiply the result by 12.0 V. We can then calculate the battery's increase in mass using $E_{\text{batt}} = (\Delta m)c^2.$ Part (b) is a simple ratio converted into a percentage.

Solution for (a)

a. Identify the knowns: $I \cdot t = 600\ \mathrm{A \cdot h};\ V = 12.0\ \mathrm{V};\ c = 3.00 \times 10^8\ \mathrm{m/s}.$

b. Identify the unknown: $\Delta m.$

c. Express the answer as an equation:

$$\begin{aligned} E_{\text{batt}} &= (\Delta m)c^2 \\ \Delta m &= \frac{E_{\text{batt}}}{c^2} \\ &= \frac{qV}{c^2} \\ &= \frac{(It)V}{c^2}. \end{aligned}$$

d. Do the calculation:

$$\Delta m = \frac{(600\ \mathrm{A \cdot h})(12.0\ \mathrm{V})}{(3.00 \times 10^8)^2}.$$

Write amperes A as coulombs per second (C/s), and convert hours into seconds:

$$\begin{aligned} \Delta m &= \frac{(600\ \mathrm{C/s \cdot h})\left(\frac{3600\ \mathrm{s}}{1\ \mathrm{h}}\right)(12.0\ \mathrm{J/C})}{(3.00 \times 10^8\ \mathrm{m/s})^2} \\ &= 2.88 \times 10^{-10}\ \mathrm{kg}. \end{aligned}$$

where we have used the conversion $1\ \mathrm{kg \cdot m^2/s^2} = 1\ \mathrm{J}.$

Solution for (b)

For part (b):

a. Identify the knowns: $\Delta m = 2.88 \times 10^{-10}\ \mathrm{kg};\ m = 20.0\ \mathrm{kg}.$

b. Identify the unknown: % change.

c. Express the answer as an equation: $\%\ \text{increase} = \frac{\Delta m}{m} \times 100\%.$

d. Do the calculation:

$$\begin{aligned}\%\text{ increase} &= \frac{\Delta m}{m} \times 100\% \\ &= \frac{2.88 \times 10^{-10}\text{ kg}}{20.0\text{ kg}} \times 100\% \\ &= 1.44 \times 10^{-9}\,\%.\end{aligned}$$

Significance

Both the actual increase in mass and the percent increase are very small, because energy is divided by c^2, a very large number. We would have to be able to measure the mass of the battery to a precision of a billionth of a percent, or 1 part in 10^{11}, to notice this increase. It is no wonder that the mass variation is not readily observed. In fact, this change in mass is so small that we may question how anyone could verify that it is real. The answer is found in nuclear processes in which the percentage of mass destroyed is large enough to be measured accurately. The mass of the fuel of a nuclear reactor, for example, is measurably smaller when its energy has been used. In that case, stored energy has been released (converted mostly into thermal energy to power electric generators) and the rest mass has decreased. A decrease in mass also occurs from using the energy stored in a battery, except that the stored energy is much greater in nuclear processes, making the change in mass measurable in practice as well as in theory.

Relativistic Energy and Momentum

We know classically that kinetic energy and momentum are related to each other, because:

$$K_{\text{class}} = \frac{p^2}{2m} = \frac{(mu)^2}{2m} = \frac{1}{2}mu^2.$$

Relativistically, we can obtain a relationship between energy and momentum by algebraically manipulating their defining equations. This yields:

$$E^2 = (pc)^2 + (mc^2)^2, \tag{5.11}$$

where E is the relativistic total energy, $E = mc^2/\sqrt{1 - u^2/c^2}$, and p is the relativistic momentum. This relationship between relativistic energy and relativistic momentum is more complicated than the classical version, but we can gain some interesting new insights by examining it. First, total energy is related to momentum and rest mass. At rest, momentum is zero, and the equation gives the total energy to be the rest energy mc^2 (so this equation is consistent with the discussion of rest energy above). However, as the mass is accelerated, its momentum p increases, thus increasing the total energy. At sufficiently high velocities, the rest energy term $(mc^2)^2$ becomes negligible compared with the momentum term $(pc)^2$; thus, $E = pc$ at extremely relativistic velocities.

If we consider momentum p to be distinct from mass, we can determine the implications of the equation $E^2 = (pc)^2 + (mc^2)^2$, for a particle that has no mass. If we take m to be zero in this equation, then $E = pc$, or $p = E/c$. Massless particles have this momentum. There are several massless particles found in nature, including photons (which are packets of electromagnetic radiation). Another implication is that a massless particle must travel at speed c and only at speed c. It is beyond the scope of this text to examine the relationship in the equation $E^2 = (pc)^2 + (mc^2)^2$ in detail, but you can see that the relationship has important implications in special relativity.

5.9 Check Your Understanding What is the kinetic energy of an electron if its speed is $0.992c$?

CHAPTER 5 REVIEW

KEY TERMS

classical (Galilean) velocity addition method of adding velocities when $v << c$; velocities add like regular numbers in one-dimensional motion: $u = v + u'$, where v is the velocity between two observers, u is the velocity of an object relative to one observer, and u' is the velocity relative to the other observer

event occurrence in space and time specified by its position and time coordinates (x, y, z, t) measured relative to a frame of reference

first postulate of special relativity laws of physics are the same in all inertial frames of reference

Galilean relativity if an observer measures a velocity in one frame of reference, and that frame of reference is moving with a velocity past a second reference frame, an observer in the second frame measures the original velocity as the vector sum of these velocities

Galilean transformation relation between position and time coordinates of the same events as seen in different reference frames, according to classical mechanics

inertial frame of reference reference frame in which a body at rest remains at rest and a body in motion moves at a constant speed in a straight line unless acted on by an outside force

length contraction decrease in observed length of an object from its proper length L_0 to length L when its length is observed in a reference frame where it is traveling at speed v

Lorentz transformation relation between position and time coordinates of the same events as seen in different reference frames, according to the special theory of relativity

Michelson-Morley experiment investigation performed in 1887 that showed that the speed of light in a vacuum is the same in all frames of reference from which it is viewed

proper length L_0; the distance between two points measured by an observer who is at rest relative to both of the points; for example, earthbound observers measure proper length when measuring the distance between two points that are stationary relative to Earth

proper time $\Delta\tau$ is the time interval measured by an observer who sees the beginning and end of the process that the time interval measures occur at the same location

relativistic kinetic energy kinetic energy of an object moving at relativistic speeds

relativistic momentum $\vec{\mathbf{p}}$, the momentum of an object moving at relativistic velocity; $\vec{\mathbf{p}} = \gamma m \vec{\mathbf{u}}$

relativistic velocity addition method of adding velocities of an object moving at a relativistic speeds

rest energy energy stored in an object at rest: $E_0 = mc^2$

rest frame frame of reference in which the observer is at rest

rest mass mass of an object as measured by an observer at rest relative to the object

second postulate of special relativity light travels in a vacuum with the same speed c in any direction in all inertial frames

special theory of relativity theory that Albert Einstein proposed in 1905 that assumes all the laws of physics have the same form in every inertial frame of reference, and that the speed of light is the same within all inertial frames

speed of light ultimate speed limit for any particle having mass

time dilation lengthening of the time interval between two events when seen in a moving inertial frame rather than the rest frame of the events (in which the events occur at the same location)

total energy sum of all energies for a particle, including rest energy and kinetic energy, given for a particle of mass m and speed u by $E = \gamma mc^2$, where $\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$

world line path through space-time

KEY EQUATIONS

Time dilation	$\Delta t = \frac{\Delta \tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \tau$
Lorentz factor	$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$
Length contraction	$L = L_0 \sqrt{1 - \frac{v^2}{c^2}} = \frac{L_0}{\gamma}$
Galilean transformation	$x = x' + vt, \quad y = y', \quad z = z', \quad t = t'$
Lorentz transformation	$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$
	$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}$
	$y = y'$
	$z = z'$
Inverse Lorentz transformation	$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$
	$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$
	$y' = y$
	$z' = z$
Space-time invariants	$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - c^2 (\Delta t)^2$
	$(\Delta \tau)^2 = -(\Delta s)^2/c^2 = (\Delta t)^2 - \frac{\left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right]}{c^2}$
Relativistic velocity addition	$u_x = \left(\frac{u'_x + v}{1 + vu'_x/c^2}\right), \quad u_y = \left(\frac{u'_y/\gamma}{1 + vu'_x/c^2}\right), \quad u_z = \left(\frac{u'_z/\gamma}{1 + vu'_x/c^2}\right)$
Relativistic Doppler effect for wavelength	$\lambda_{obs} = \lambda_s \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$
Relativistic Doppler effect for frequency	$f_{obs} = f_s \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}$

Relativistic momentum	$\vec{\mathbf{p}} = \gamma m \vec{\mathbf{u}} = \frac{m \vec{\mathbf{u}}}{\sqrt{1 - \frac{u^2}{c}}}$
Relativistic total energy	$E = \gamma mc^2, \text{ where } \gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$
Relativistic kinetic energy	$K_{\text{rel}} = (\gamma - 1)mc^2, \text{ where } \gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$

SUMMARY

5.1 Invariance of Physical Laws

- Relativity is the study of how observers in different reference frames measure the same event.
- Modern relativity is divided into two parts. Special relativity deals with observers in uniform (unaccelerated) motion, whereas general relativity includes accelerated relative motion and gravity. Modern relativity is consistent with all empirical evidence thus far and, in the limit of low velocity and weak gravitation, gives close agreement with the predictions of classical (Galilean) relativity.
- An inertial frame of reference is a reference frame in which a body at rest remains at rest and a body in motion moves at a constant speed in a straight line unless acted upon by an outside force.
- Modern relativity is based on Einstein's two postulates. The first postulate of special relativity is that the laws of physics are the same in all inertial frames of reference. The second postulate of special relativity is that the speed of light c is the same in all inertial frames of reference, independent of the relative motion of the observer and the light source.
- The Michelson-Morley experiment demonstrated that the speed of light in a vacuum is independent of the motion of Earth about the sun.

5.2 Relativity of Simultaneity

- Two events are defined to be simultaneous if an observer measures them as occurring at the same time (such as by receiving light from the events).
- Two events at locations a distance apart that are simultaneous for an observer at rest in one frame of reference are not necessarily simultaneous for an observer at rest in a different frame of reference.

5.3 Time Dilation

- Two events are defined to be simultaneous if an observer measures them as occurring at the same time. They are not necessarily simultaneous to all observers—simultaneity is not absolute.
- Time dilation is the lengthening of the time interval between two events when seen in a moving inertial frame rather than the rest frame of the events (in which the events occur at the same location).
- Observers moving at a relative velocity v do not measure the same elapsed time between two events. Proper time $\Delta\tau$ is the time measured in the reference frame where the start and end of the time interval occur at the same location. The time interval Δt measured by an observer who sees the frame of events moving at speed v is related to the proper time interval $\Delta\tau$ of the events by the equation:

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma\Delta\tau,$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

- The premise of the twin paradox is faulty because the traveling twin is accelerating. The journey is not symmetrical for the two twins.
- Time dilation is usually negligible at low relative velocities, but it does occur, and it has been verified by experiment.
- The proper time is the shortest measure of any time interval. Any observer who is moving relative to the system being observed measures a time interval longer than the proper time.

5.4 Length Contraction

- All observers agree upon relative speed.
- Distance depends on an observer's motion. Proper length L_0 is the distance between two points measured by an observer who is at rest relative to both of the points.
- Length contraction is the decrease in observed length of an object from its proper length L_0 to length L when its length is observed in a reference frame where it is traveling at speed v.
- The proper length is the longest measurement of any length interval. Any observer who is moving relative to the system being observed measures a length shorter than the proper length.

5.5 The Lorentz Transformation

- The Galilean transformation equations describe how, in classical nonrelativistic mechanics, the position, velocity, and accelerations measured in one frame appear in another. Lengths remain unchanged and a single universal time scale is assumed to apply to all inertial frames.
- Newton's laws of mechanics obey the principle of having the same form in all inertial frames under a Galilean transformation, given by

  $$x = x' + vt, \quad y = y', \quad z = z', \quad t = t'.$$

  The concept that times and distances are the same in all inertial frames in the Galilean transformation, however, is inconsistent with the postulates of special relativity.
- The relativistically correct Lorentz transformation equations are

Lorentz transformation	Inverse Lorentz transformation
$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$	$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$
$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}$	$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$
$y = y'$	$y' = y$
$z = z'$	$z' = z$

  We can obtain these equations by requiring an expanding spherical light signal to have the same shape and speed of growth, c, in both reference frames.
- Relativistic phenomena can be explained in terms of the geometrical properties of four-dimensional space-time, in which Lorentz transformations correspond to rotations of axes.
- The Lorentz transformation corresponds to a space-time axis rotation, similar in some ways to a rotation of space axes, but in which the invariant spatial separation is given by Δs rather than distances Δr, and that the Lorentz transformation involving the time axis does not preserve perpendicularity of axes or the scales along the axes.

- The analysis of relativistic phenomena in terms of space-time diagrams supports the conclusion that these phenomena result from properties of space and time itself, rather than from the laws of electromagnetism.

5.6 Relativistic Velocity Transformation

- With classical velocity addition, velocities add like regular numbers in one-dimensional motion: $u = v + u'$, where v is the velocity between two observers, u is the velocity of an object relative to one observer, and u' is the velocity relative to the other observer.
- Velocities cannot add to be greater than the speed of light.
- Relativistic velocity addition describes the velocities of an object moving at a relativistic velocity.

5.7 Doppler Effect for Light

- An observer of electromagnetic radiation sees relativistic Doppler effects if the source of the radiation is moving relative to the observer. The wavelength of the radiation is longer (called a red shift) than that emitted by the source when the source moves away from the observer and shorter (called a blue shift) when the source moves toward the observer. The shifted wavelength is described by the equation:

$$\lambda_{\text{obs}} = \lambda_s \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}.$$

  where λ_{obs} is the observed wavelength, λ_s is the source wavelength, and v is the relative velocity of the source to the observer.

5.8 Relativistic Momentum

- The law of conservation of momentum is valid for relativistic momentum whenever the net external force is zero. The relativistic momentum is $p = \gamma m u$, where m is the rest mass of the object, u is its velocity relative to an observer, and the relativistic factor is $\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$.
- At low velocities, relativistic momentum is equivalent to classical momentum.
- Relativistic momentum approaches infinity as u approaches c. This implies that an object with mass cannot reach the speed of light.

5.9 Relativistic Energy

- The relativistic work-energy theorem is $W_{\text{net}} = E - E_0 = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2$.
- Relativistically, $W_{\text{net}} = K_{\text{rel}}$ where K_{rel} is the relativistic kinetic energy.
- An object of *mass m* at velocity u has kinetic energy $K_{\text{rel}} = (\gamma - 1)mc^2$, where $\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$.
- At low velocities, relativistic kinetic energy reduces to classical kinetic energy.
- No object with mass can attain the speed of light, because an infinite amount of work and an infinite amount of energy input is required to accelerate a mass to the speed of light.
- Relativistic energy is conserved as long as we define it to include the possibility of mass changing to energy.
- The total energy of a particle with mass m traveling at speed u is defined as $E = \gamma mc^2$, where $\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$ and u denotes the velocity of the particle.

- The rest energy of an object of mass *m* is $E_0 = mc^2,$ meaning that mass is a form of energy. If energy is stored in an object, its mass increases. Mass can be destroyed to release energy.
- We do not ordinarily notice the increase or decrease in mass of an object because the change in mass is so small for a large increase in energy. The equation $E^2 = (pc)^2 + (mc^2)^2$ relates the relativistic total energy *E* and the relativistic momentum *p*. At extremely high velocities, the rest energy mc^2 becomes negligible, and $E = pc$.

CONCEPTUAL QUESTIONS

5.1 Invariance of Physical Laws

1. Which of Einstein's postulates of special relativity includes a concept that does not fit with the ideas of classical physics? Explain.

2. Is Earth an inertial frame of reference? Is the sun? Justify your response.

3. When you are flying in a commercial jet, it may appear to you that the airplane is stationary and Earth is moving beneath you. Is this point of view valid? Discuss briefly.

5.3 Time Dilation

4. (a) Does motion affect the rate of a clock as measured by an observer moving with it? (b) Does motion affect how an observer moving relative to a clock measures its rate?

5. To whom does the elapsed time for a process seem to be longer, an observer moving relative to the process or an observer moving with the process? Which observer measures the interval of proper time?

6. (a) How could you travel far into the future of Earth without aging significantly? (b) Could this method also allow you to travel into the past?

5.4 Length Contraction

7. To whom does an object seem greater in length, an observer moving with the object or an observer moving relative to the object? Which observer measures the object's proper length?

8. Relativistic effects such as time dilation and length contraction are present for cars and airplanes. Why do these effects seem strange to us?

9. Suppose an astronaut is moving relative to Earth at a significant fraction of the speed of light. (a) Does he observe the rate of his clocks to have slowed? (b) What change in the rate of earthbound clocks does he see? (c) Does his ship seem to him to shorten? (d) What about the distance between two stars that lie in the direction of his motion? (e) Do he and an earthbound observer agree on his velocity relative to Earth?

5.7 Doppler Effect for Light

10. Explain the meaning of the terms "red shift" and "blue shift" as they relate to the relativistic Doppler effect.

11. What happens to the relativistic Doppler effect when relative velocity is zero? Is this the expected result?

12. Is the relativistic Doppler effect consistent with the classical Doppler effect in the respect that λ_{obs} is larger for motion away?

13. All galaxies farther away than about 50×10^6 ly exhibit a red shift in their emitted light that is proportional to distance, with those farther and farther away having progressively greater red shifts. What does this imply, assuming that the only source of red shift is relative motion?

5.8 Relativistic Momentum

14. How does modern relativity modify the law of conservation of momentum?

15. Is it possible for an external force to be acting on a system and relativistic momentum to be conserved? Explain.

5.9 Relativistic Energy

16. How are the classical laws of conservation of energy and conservation of mass modified by modern relativity?

17. What happens to the mass of water in a pot when it cools, assuming no molecules escape or are added? Is this observable in practice? Explain.

18. Consider a thought experiment. You place an expanded balloon of air on weighing scales outside in the early morning. The balloon stays on the scales and you are able to measure changes in its mass. Does the mass of the balloon change as the day progresses? Discuss the difficulties in carrying out this experiment.

19. The mass of the fuel in a nuclear reactor decreases by an observable amount as it puts out energy. Is the same true for the coal and oxygen combined in a conventional power plant? If so, is this observable in practice for the coal and oxygen? Explain.

20. We know that the velocity of an object with mass has an upper limit of c. Is there an upper limit on its momentum? Its energy? Explain.

21. Given the fact that light travels at c, can it have mass? Explain.

22. If you use an Earth-based telescope to project a laser beam onto the moon, you can move the spot across the moon's surface at a velocity greater than the speed of light. Does this violate modern relativity? (Note that light is being sent from the Earth to the moon, not across the surface of the moon.)

PROBLEMS

5.3 Time Dilation

23. (a) What is γ if $v = 0.250c$? (b) If $v = 0.500c$?

24. (a) What is γ if $v = 0.100c$? (b) If $v = 0.900c$?

25. Particles called π-mesons are produced by accelerator beams. If these particles travel at 2.70×10^8 m/s and live 2.60×10^{-8} s when at rest relative to an observer, how long do they live as viewed in the laboratory?

26. Suppose a particle called a kaon is created by cosmic radiation striking the atmosphere. It moves by you at $0.980c$, and it lives 1.24×10^{-8} s when at rest relative to an observer. How long does it live as you observe it?

27. A neutral π-meson is a particle that can be created by accelerator beams. If one such particle lives 1.40×10^{-16} s as measured in the laboratory, and 0.840×10^{-16} s when at rest relative to an observer, what is its velocity relative to the laboratory?

28. A neutron lives 900 s when at rest relative to an observer. How fast is the neutron moving relative to an observer who measures its life span to be 2065 s?

29. If relativistic effects are to be less than 1%, then γ must be less than 1.01. At what relative velocity is $\gamma = 1.01$?

30. If relativistic effects are to be less than 3%, then γ must be less than 1.03. At what relative velocity is $\gamma = 1.03$?

5.4 Length Contraction

31. A spaceship, 200 m long as seen on board, moves by the Earth at $0.970c$. What is its length as measured by an earthbound observer?

32. How fast would a 6.0 m-long sports car have to be going past you in order for it to appear only 5.5 m long?

33. (a) How far does the muon in **Example 5.1** travel according to the earthbound observer? (b) How far does it travel as viewed by an observer moving with it? Base your calculation on its velocity relative to the Earth and the time it lives (proper time). (c) Verify that these two distances are related through length contraction $\gamma = 3.20$.

34. (a) How long would the muon in **Example 5.1** have lived as observed on Earth if its velocity was $0.0500c$? (b) How far would it have traveled as observed on Earth? (c) What distance is this in the muon's frame?

35. **Unreasonable Results** A spaceship is heading directly toward Earth at a velocity of $0.800c$. The astronaut on board claims that he can send a canister toward the Earth at $1.20c$ relative to Earth. (a) Calculate the velocity the canister must have relative to the spaceship. (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

5.5 The Lorentz Transformation

36. Describe the following physical occurrences as events, that is, in the form (x, y, z, t): (a) A postman rings a doorbell of a house precisely at noon. (b) At the same time as the doorbell is rung, a slice of bread pops out of a toaster that is located 10 m from the door in the east direction from the door. (c) Ten seconds later, an airplane arrives at the airport, which is 10 km from the door in the east direction and 2 km to the south.

37. Describe what happens to the angle $\alpha = \tan(v/c)$, and therefore to the transformed axes in **Figure 5.17**, as the relative velocity v of the S and S' frames of reference approaches c.

38. Describe the shape of the world line on a space-time diagram of (a) an object that remains at rest at a specific position along the x-axis; (b) an object that moves at constant velocity u in the x-direction; (c) an object that begins at rest and accelerates at a constant rate of in the positive x-direction.

39. A man standing still at a train station watches two boys throwing a baseball in a moving train. Suppose the train is moving east with a constant speed of 20 m/s and one of the boys throws the ball with a speed of 5 m/s with respect to himself toward the other boy, who is 5 m west from him. What is the velocity of the ball as observed by the man on the station?

40. When observed from the sun at a particular instant, Earth and Mars appear to move in opposite directions with speeds 108,000 km/h and 86,871 km/h, respectively. What is the speed of Mars at this instant when observed from Earth?

41. A man is running on a straight road perpendicular to a train track and away from the track at a speed of 12 m/s. The train is moving with a speed of 30 m/s with respect to the track. What is the speed of the man with respect to a passenger sitting at rest in the train?

42. A man is running on a straight road that makes $30°$ with the train track. The man is running in the direction on the road that is away from the track at a speed of 12 m/s. The train is moving with a speed of 30 m/s with respect to the track. What is the speed of the man with respect to a passenger sitting at rest in the train?

43. In a frame at rest with respect to the billiard table, a billiard ball of mass m moving with speed v strikes another billiard ball of mass m at rest. The first ball comes to rest after the collision while the second ball takes off with speed v in the original direction of the motion of the first ball. This shows that momentum is conserved in this frame. (a) Now, describe the same collision from the perspective of a frame that is moving with speed v in the direction of the motion of the first ball. (b) Is the momentum conserved in this frame?

44. In a frame at rest with respect to the billiard table, two billiard balls of same mass m are moving toward each other with the same speed v. After the collision, the two balls come to rest. (a) Show that momentum is conserved in this frame. (b) Now, describe the same collision from the perspective of a frame that is moving with speed v in the direction of the motion of the first ball. (c) Is the momentum conserved in this frame?

45. In a frame S, two events are observed: event 1: a pion is created at rest at the origin and event 2: the pion disintegrates after time τ. Another observer in a frame S' is moving in the positive direction along the positive x-axis with a constant speed v and observes the same two events in his frame. The origins of the two frames coincide at $t = t' = 0$. (a) Find the positions and timings of these two events in the frame S' (a) according to the Galilean transformation, and (b) according to the Lorentz transformation.

5.6 Relativistic Velocity Transformation

46. If two spaceships are heading directly toward each other at 0.800c, at what speed must a canister be shot from the first ship to approach the other at 0.999c as seen by the second ship?

47. Two planets are on a collision course, heading directly toward each other at 0.250c. A spaceship sent from one planet approaches the second at 0.750c as seen by the second planet. What is the velocity of the ship relative to the first planet?

48. When a missile is shot from one spaceship toward another, it leaves the first at 0.950c and approaches the other at 0.750c. What is the relative velocity of the two ships?

49. What is the relative velocity of two spaceships if one fires a missile at the other at 0.750c and the other observes it to approach at 0.950c?

50. Prove that for any relative velocity v between two observers, a beam of light sent from one to the other will approach at speed c (provided that v is less than c, of course).

51. Show that for any relative velocity v between two observers, a beam of light projected by one directly away from the other will move away at the speed of light (provided that v is less than c, of course).

5.7 Doppler Effect for Light

52. A highway patrol officer uses a device that measures the speed of vehicles by bouncing radar off them and measuring the Doppler shift. The outgoing radar has a frequency of 100 GHz and the returning echo has a frequency 15.0 kHz higher. What is the velocity of the vehicle? Note that there are two Doppler shifts in echoes. Be certain not to round off until the end of the problem, because the effect is small.

5.8 Relativistic Momentum

53. Find the momentum of a helium nucleus having a mass of 6.68×10^{-27} kg that is moving at 0.200c.

54. What is the momentum of an electron traveling at 0.980c?

55. (a) Find the momentum of a 1.00×10^{9}-kg asteroid heading towards Earth at 30.0 km/s. (b) Find the ratio of this momentum to the classical momentum. (Hint: Use the approximation that $\gamma = 1 + (1/2)v^2/c^2$ at low velocities.)

56. (a) What is the momentum of a 2000-kg satellite orbiting at 4.00 km/s? (b) Find the ratio of this momentum to the classical momentum. (Hint: Use the approximation that $\gamma = 1 + (1/2)v^2/c^2$ at low velocities.)

57. What is the velocity of an electron that has a momentum of 3.04×10^{-21} kg · m/s ? Note that you must calculate the velocity to at least four digits to see the difference from c.

58. Find the velocity of a proton that has a momentum of 4.48×10^{-19} kg · m/s.

5.9 Relativistic Energy

59. What is the rest energy of an electron, given its mass is 9.11×10^{-31} kg? Give your answer in joules and MeV.

60. Find the rest energy in joules and MeV of a proton, given its mass is 1.67×10^{-27} kg.

61. If the rest energies of a proton and a neutron (the two constituents of nuclei) are 938.3 and 939.6 MeV, respectively, what is the difference in their mass in kilograms?

62. The Big Bang that began the universe is estimated to have released 10^{68} J of energy. How many stars could half this energy create, assuming the average star's mass is 4.00×10^{30} kg ?

63. A supernova explosion of a 2.00×10^{31} kg star produces 1.00×10^{44} J of energy. (a) How many kilograms of mass are converted to energy in the explosion? (b) What is the ratio $\Delta m/m$ of mass destroyed to the original mass of the star?

64. (a) Using data from **m58312 (http://cnx.org/content/m58312/latest/#fs-id1165039443587)**, calculate the mass converted to energy by the fission of 1.00 kg of uranium. (b) What is the ratio of mass destroyed to the original mass, $\Delta m/m$?

65. (a) Using data from **m58312 (http://cnx.org/content/m58312/latest/#fs-id1165039443587)**, calculate the amount of mass converted to energy by the fusion of 1.00 kg of hydrogen. (b) What is the ratio of mass destroyed to the original mass, $\Delta m/m$? (c) How does this compare with $\Delta m/m$ for the fission of 1.00 kg of uranium?

66. There is approximately 10^{34} J of energy available from fusion of hydrogen in the world's oceans. (a) If 10^{33} J of this energy were utilized, what would be the decrease in mass of the oceans? (b) How great a volume of water does this correspond to? (c) Comment on whether this is a significant fraction of the total mass of the oceans.

67. A muon has a rest mass energy of 105.7 MeV, and it decays into an electron and a massless particle. (a) If all the lost mass is converted into the electron's kinetic energy, find γ for the electron. (b) What is the electron's velocity?

68. A π-meson is a particle that decays into a muon and a massless particle. The π-meson has a rest mass energy of 139.6 MeV, and the muon has a rest mass energy of 105.7 MeV. Suppose the π-meson is at rest and all of the missing mass goes into the muon's kinetic energy. How fast will the muon move?

69. (a) Calculate the relativistic kinetic energy of a 1000-kg car moving at 30.0 m/s if the speed of light were only 45.0 m/s. (b) Find the ratio of the relativistic kinetic energy to classical.

70. Alpha decay is nuclear decay in which a helium nucleus is emitted. If the helium nucleus has a mass of 6.80×10^{-27} kg and is given 5.00 MeV of kinetic energy, what is its velocity?

71. (a) Beta decay is nuclear decay in which an electron is emitted. If the electron is given 0.750 MeV of kinetic energy, what is its velocity? (b) Comment on how the high velocity is consistent with the kinetic energy as it compares to the rest mass energy of the electron.

ADDITIONAL PROBLEMS

72. (a) At what relative velocity is $\gamma = 1.50$? (b) At what relative velocity is $\gamma = 100$?

73. (a) At what relative velocity is $\gamma = 2.00$? (b) At what relative velocity is $\gamma = 10.0$?

74. **Unreasonable Results** (a) Find the value of γ required for the following situation. An earthbound observer measures 23.9 h to have passed while signals from a high-velocity space probe indicate that 24.0 h have passed on board. (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

75. (a) How long does it take the astronaut in Example 5.5 to travel 4.30 ly at $0.99944c$ (as measured by the earthbound observer)? (b) How long does it take according to the astronaut? (c) Verify that these two times are related through time dilation with $\gamma = 30.00$ as given.

76. (a) How fast would an athlete need to be running for a 100- m race to look 100 yd long? (b) Is the answer consistent with the fact that relativistic effects are difficult to observe in ordinary circumstances? Explain.

77. (a) Find the value of γ for the following situation. An astronaut measures the length of his spaceship to be 100 m, while an earthbound observer measures it to be 25.0 m. (b) What is the speed of the spaceship relative to Earth?

78. A clock in a spaceship runs one-tenth the rate at which an identical clock on Earth runs. What is the speed of the spaceship?

79. An astronaut has a heartbeat rate of 66 beats per minute as measured during his physical exam on Earth. The heartbeat rate of the astronaut is measured when he is in a spaceship traveling at 0.5*c* with respect to Earth by an observer (A) in the ship and by an observer (B) on Earth. (a) Describe an experimental method by which observer B on Earth will be able to determine the heartbeat rate of the astronaut when the astronaut is in the spaceship. (b) What will be the heartbeat rate(s) of the astronaut reported by observers A and B?

80. A spaceship (A) is moving at speed *c*/2 with respect to another spaceship (B). Observers in A and B set their clocks so that the event at (*x*, *y*, *z*, *t*) of turning on a laser in spaceship B has coordinates (0, 0, 0, 0) in A and also (0, 0, 0, 0) in B. An observer at the origin of B turns on the laser at $t = 0$ and turns it off at $t = \tau$ in his time. What is the time duration between on and off as seen by an observer in A?

81. Same two observers as in the preceding exercise, but now we look at two events occurring in spaceship A. A photon arrives at the origin of A at its time $t = 0$ and another photon arrives at $(x = 1.00\text{ m},\ 0,\ 0)$ at $t = 0$ in the frame of ship A. (a) Find the coordinates and times of the two events as seen by an observer in frame B. (b) In which frame are the two events simultaneous and in which frame are they not simultaneous?

82. Same two observers as in the preceding exercises. A rod of length 1 m is laid out on the *x*-axis in the frame of B from origin to $(x = 1.00\text{ m},\ 0,\ 0)$. What is the length of the rod observed by an observer in the frame of spaceship A?

83. An observer at origin of inertial frame S sees a flashbulb go off at $x = 150\text{ km},\ y = 15.0\text{ km},$ and $z = 1.00\text{ km}$ at time $t = 4.5 \times 10^{-4}\text{ s}$. At what time and position in the S′ system did the flash occur, if S′ is moving along shared *x*-direction with S at a velocity $v = 0.6c$?

84. An observer sees two events 1.5×10^{-8} s apart at a separation of 800 m. How fast must a second observer be moving relative to the first to see the two events occur simultaneously?

85. An observer standing by the railroad tracks sees two bolts of lightning strike the ends of a 500-m-long train simultaneously at the instant the middle of the train passes him at 50 m/s. Use the Lorentz transformation to find the time between the lightning strikes as measured by a passenger seated in the middle of the train.

86. Two astronomical events are observed from Earth to occur at a time of 1 s apart and a distance separation of 1.5×10^9 m from each other. (a) Determine whether separation of the two events is space like or time like. (b) State what this implies about whether it is consistent with special relativity for one event to have caused the other?

87. Two astronomical events are observed from Earth to occur at a time of 0.30 s apart and a distance separation of 2.0×10^9 m from each other. How fast must a spacecraft travel from the site of one event toward the other to make the events occur at the same time when measured in the frame of reference of the spacecraft?

88. A spacecraft starts from being at rest at the origin and accelerates at a constant rate *g*, as seen from Earth, taken to be an inertial frame, until it reaches a speed of *c/*2. (a) Show that the increment of proper time is related to the elapsed time in Earth's frame by: $d\tau = \sqrt{1 - v^2/c^2}dt$.

(b) Find an expression for the elapsed time to reach speed *c/*2 as seen in Earth's frame. (c) Use the relationship in (a) to obtain a similar expression for the elapsed proper time to reach *c*/2 as seen in the spacecraft, and determine the ratio of the time seen from Earth with that on the spacecraft to reach the final speed.

89. (a) All but the closest galaxies are receding from our own Milky Way Galaxy. If a galaxy 12.0×10^9 ly away is receding from us at 0.900*c*, at what velocity relative to us must we send an exploratory probe to approach the other galaxy at 0.990*c* as measured from that galaxy? (b) How long will it take the probe to reach the other galaxy as measured from Earth? You may assume that the velocity of the other galaxy remains constant. (c) How long will it then take for a radio signal to be beamed back? (All of this is possible in principle, but not practical.)

90. Suppose a spaceship heading straight toward the Earth at 0.750*c* can shoot a canister at 0.500*c* relative to the ship. (a) What is the velocity of the canister relative to Earth, if it is shot directly at Earth? (b) If it is shot directly away from Earth?

91. Repeat the preceding problem with the ship heading directly away from Earth.

92. If a spaceship is approaching the Earth at 0.100*c* and a message capsule is sent toward it at 0.100*c* relative to Earth, what is the speed of the capsule relative to the ship?

93. (a) Suppose the speed of light were only 3000 m/s. A jet fighter moving toward a target on the ground at 800 m/s shoots bullets, each having a muzzle velocity of 1000 m/s. What are the bullets' velocity relative to the target? (b) If the speed of light was this small, would you observe relativistic effects in everyday life? Discuss.

94. If a galaxy moving away from the Earth has a speed of 1000 km/s and emits 656 nm light characteristic of hydrogen (the most common element in the universe). (a) What wavelength would we observe on Earth? (b) What type of electromagnetic radiation is this? (c) Why is the speed of Earth in its orbit negligible here?

95. A space probe speeding towards the nearest star moves at $0.250c$ and sends radio information at a broadcast frequency of 1.00 GHz. What frequency is received on Earth?

96. Near the center of our galaxy, hydrogen gas is moving directly away from us in its orbit about a black hole. We receive 1900 nm electromagnetic radiation and know that it was 1875 nm when emitted by the hydrogen gas. What is the speed of the gas?

97. (a) Calculate the speed of a 1.00-μg particle of dust that has the same momentum as a proton moving at 0.999*c*. (b) What does the small speed tell us about the mass of a proton compared to even a tiny amount of macroscopic matter?

98. (a) Calculate γ for a proton that has a momentum of $1.00 \text{ kg} \cdot \text{m/s}$. (b) What is its speed? Such protons form a rare component of cosmic radiation with uncertain origins.

99. Show that the relativistic form of Newton's second law is (a) $F = m\frac{du}{dt}\frac{1}{\left(1 - u^2/c^2\right)^{3/2}}$; (b) Find the force needed to accelerate a mass of 1 kg by 1 m/s^2 when it is traveling at a velocity of *c*/2.

100. A positron is an antimatter version of the electron, having exactly the same mass. When a positron and an electron meet, they annihilate, converting all of their mass into energy. (a) Find the energy released, assuming negligible kinetic energy before the annihilation. (b) If this energy is given to a proton in the form of kinetic energy, what is its velocity? (c) If this energy is given to another electron in the form of kinetic energy, what is its velocity?

101. What is the kinetic energy in MeV of a π-meson that lives 1.40×10^{-16} s as measured in the laboratory, and 0.840×10^{-16} s when at rest relative to an observer, given that its rest energy is 135 MeV?

102. Find the kinetic energy in MeV of a neutron with a measured life span of 2065 s, given its rest energy is 939.6 MeV, and rest life span is 900s.

103. (a) Show that $(pc)^2/(mc^2)^2 = \gamma^2 - 1$. This means that at large velocities $pc > > mc^2$. (b) Is $E \approx pc$ when $\gamma = 30.0$, as for the astronaut discussed in the twin paradox?

104. One cosmic ray neutron has a velocity of $0.250c$ relative to the Earth. (a) What is the neutron's total energy in MeV? (b) Find its momentum. (c) Is $E \approx pc$ in this situation? Discuss in terms of the equation given in part (a) of the previous problem.

105. What is γ for a proton having a mass energy of 938.3 MeV accelerated through an effective potential of 1.0 TV (teravolt)?

106. (a) What is the effective accelerating potential for electrons at the Stanford Linear Accelerator, if $\gamma = 1.00 \times 10^5$ for them? (b) What is their total energy (nearly the same as kinetic in this case) in GeV?

107. (a) Using data from **m58312 (http://cnx.org/content/m58312/latest/#fs-id1165039443587)** , find the mass destroyed when the energy in a barrel of crude oil is released. (b) Given these barrels contain 200 liters and assuming the density of crude oil is 750kg/m^3, what is the ratio of mass destroyed to original mass, $\Delta m/m$?

108. (a) Calculate the energy released by the destruction of 1.00 kg of mass. (b) How many kilograms could be lifted to a 10.0 km height by this amount of energy?

109. A Van de Graaff accelerator utilizes a 50.0 MV potential difference to accelerate charged particles such as protons. (a) What is the velocity of a proton accelerated by such a potential? (b) An electron?

110. Suppose you use an average of $500 \text{ kW} \cdot \text{h}$ of electric energy per month in your home. (a) How long would 1.00 g of mass converted to electric energy with an efficiency of 38.0% last you? (b) How many homes could be supplied at the $500 \text{ kW} \cdot \text{h}$ per month rate for one year by the energy from the described mass conversion?

111. (a) A nuclear power plant converts energy from nuclear fission into electricity with an efficiency of 35.0%. How much mass is destroyed in one year to produce a continuous 1000 MW of electric power? (b) Do you think it would be possible to observe this mass loss if the total mass of the fuel is 10^4 kg?

112. Nuclear-powered rockets were researched for some years before safety concerns became paramount. (a) What fraction of a rocket's mass would have to be destroyed to get it into a low Earth orbit, neglecting the decrease in gravity? (Assume an orbital altitude of 250 km, and calculate both the kinetic energy (classical) and the gravitational potential energy needed.) (b) If the ship has a mass of 1.00×10^5 kg (100 tons), what total yield nuclear explosion in tons of TNT is needed?

113. The sun produces energy at a rate of 3.85×10^{26} W by the fusion of hydrogen. About 0.7% of each kilogram of hydrogen goes into the energy generated by the Sun. (a) How many kilograms of hydrogen undergo fusion each second? (b) If the sun is 90.0% hydrogen and half of this can undergo fusion before the sun changes character, how long could it produce energy at its current rate? (c) How many kilograms of mass is the sun losing per second? (d) What fraction of its mass will it have lost in the time found in part (b)?

114. Show that $E^2 - p^2c^2$ for a particle is invariant under Lorentz transformations.

6 | PHOTONS AND MATTER WAVES

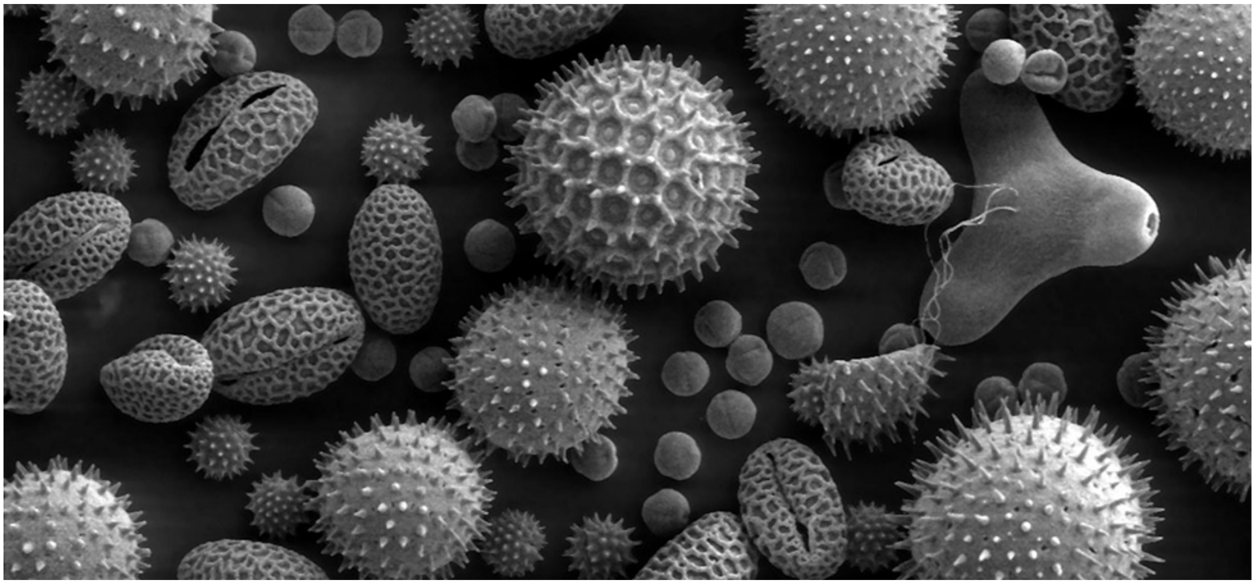


Figure 6.1 In this image of pollen taken with an electron microscope, the bean-shaped grains are about $50 \mu m$ long. Electron microscopes can have a much higher resolving power than a conventional light microscope because electron wavelengths can be 100,000 times shorter than the wavelengths of visible-light photons. (credit: modification of work by Dartmouth College Electron Microscope Facility)

Chapter Outline

Introduction

Two of the most revolutionary concepts of the twentieth century were the description of light as a collection of particles, and the treatment of particles as waves. These wave properties of matter have led to the discovery of technologies such as electron microscopy, which allows us to examine submicroscopic objects such as grains of pollen, as shown above.

In this chapter, you will learn about the energy quantum, a concept that was introduced in 1900 by the German physicist Max Planck to explain blackbody radiation. We discuss how Albert Einstein extended Planck's concept to a quantum of light (a "photon") to explain the photoelectric effect. We also show how American physicist Arthur H. Compton used the photon concept in 1923 to explain wavelength shifts observed in X-rays. After a discussion of Bohr's model of hydrogen, we describe how matter waves were postulated in 1924 by Louis-Victor de Broglie to justify Bohr's model and we examine the experiments conducted in 1923–1927 by Clinton Davisson and Lester Germer that confirmed the existence of de Broglie's matter waves.

6.1 | Blackbody Radiation

Learning Objectives

By the end of this section you will be able to:

- Apply Wien's and Stefan's laws to analyze radiation emitted by a blackbody
- Explain Planck's hypothesis of energy quanta

All bodies emit electromagnetic radiation over a range of wavelengths. In an earlier chapter, we learned that a cooler body radiates less energy than a warmer body. We also know by observation that when a body is heated and its temperature rises, the perceived wavelength of its emitted radiation changes from infrared to red, and then from red to orange, and so forth. As its temperature rises, the body glows with the colors corresponding to ever-smaller wavelengths of the electromagnetic spectrum. This is the underlying principle of the incandescent light bulb: A hot metal filament glows red, and when heating continues, its glow eventually covers the entire visible portion of the electromagnetic spectrum. The temperature (T) of the object that emits radiation, or the **emitter**, determines the wavelength at which the radiated energy is at its maximum. For example, the Sun, whose surface temperature is in the range between 5000 K and 6000 K, radiates most strongly in a range of wavelengths about 560 nm in the visible part of the electromagnetic spectrum. Your body, when at its normal temperature of about 300 K, radiates most strongly in the infrared part of the spectrum.

Radiation that is incident on an object is partially absorbed and partially reflected. At thermodynamic equilibrium, the rate at which an object absorbs radiation is the same as the rate at which it emits it. Therefore, a good **absorber** of radiation (any object that absorbs radiation) is also a good emitter. A perfect absorber absorbs all electromagnetic radiation incident on it; such an object is called a **blackbody**.

Although the blackbody is an idealization, because no physical object absorbs 100% of incident radiation, we can construct a close realization of a blackbody in the form of a small hole in the wall of a sealed enclosure known as a cavity radiator, as shown in **Figure 6.2**. The inside walls of a cavity radiator are rough and blackened so that any radiation that enters through a tiny hole in the cavity wall becomes trapped inside the cavity. At thermodynamic equilibrium (at temperature T), the cavity walls absorb exactly as much radiation as they emit. Furthermore, inside the cavity, the radiation entering the hole is balanced by the radiation leaving it. The emission spectrum of a blackbody can be obtained by analyzing the light radiating from the hole. Electromagnetic waves emitted by a blackbody are called **blackbody radiation**.

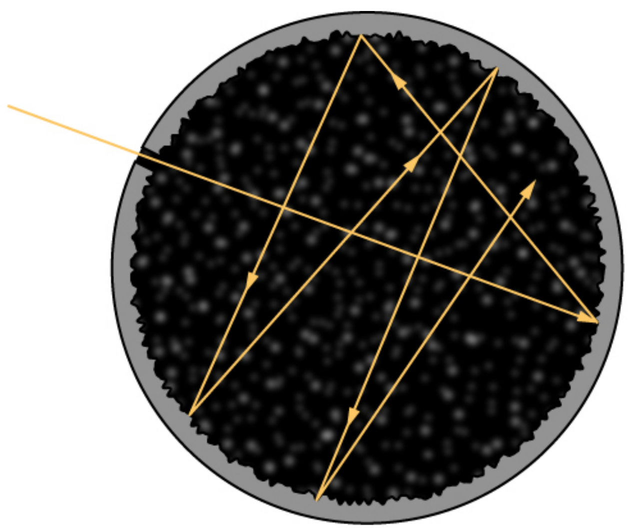


Figure 6.2 A blackbody is physically realized by a small hole in the wall of a cavity radiator.

The intensity $I(\lambda, T)$ of blackbody radiation depends on the wavelength λ of the emitted radiation and on the temperature T of the blackbody (**Figure 6.3**). The function $I(\lambda, T)$ is the **power intensity** that is radiated per unit wavelength; in other words, it is the power radiated per unit area of the hole in a cavity radiator per unit wavelength. According to this definition, $I(\lambda, T)d\lambda$ is the power per unit area that is emitted in the wavelength interval from λ to $\lambda + d\lambda$. The intensity distribution among wavelengths of radiation emitted by cavities was studied experimentally at the end of the nineteenth century. Generally, radiation emitted by materials only approximately follows the blackbody radiation curve (**Figure 6.4**); however, spectra of common stars do follow the blackbody radiation curve very closely.

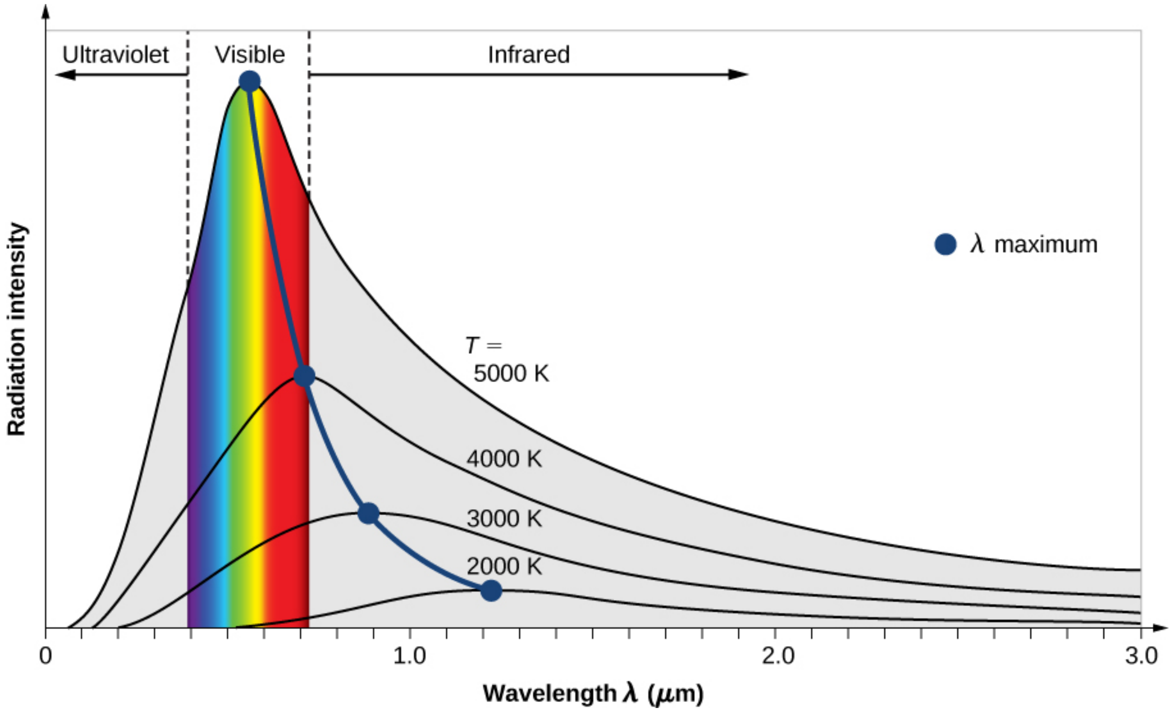


Figure 6.3 The intensity of blackbody radiation versus the wavelength of the emitted radiation. Each curve corresponds to a different blackbody temperature, starting with a low temperature (the lowest curve) to a high temperature (the highest curve).

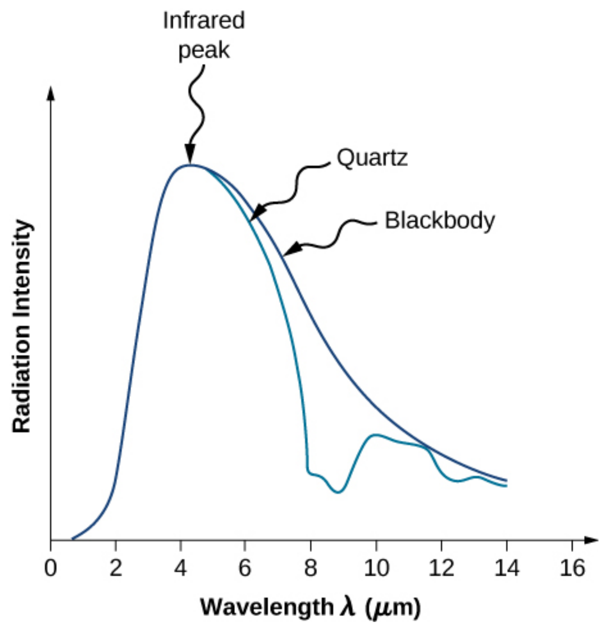


Figure 6.4 The spectrum of radiation emitted from a quartz surface (blue curve) and the blackbody radiation curve (black curve) at 600 K.

Two important laws summarize the experimental findings of blackbody radiation: *Wien's displacement law* and *Stefan's law*. Wien's displacement law is illustrated in **Figure 6.3** by the curve connecting the maxima on the intensity curves. In these curves, we see that the hotter the body, the shorter the wavelength corresponding to the emission peak in the radiation curve. Quantitatively, Wien's law reads

$$\lambda_{\text{max}} T = 2.898 \times 10^{-3}\,\text{m} \cdot \text{K} \tag{6.1}$$

where λ_{max} is the position of the maximum in the radiation curve. In other words, λ_{max} is the wavelength at which a blackbody radiates most strongly at a given temperature T. Note that in **Equation 6.1**, the temperature is in kelvins. Wien's displacement law allows us to estimate the temperatures of distant stars by measuring the wavelength of radiation they emit.

Example 6.1

Temperatures of Distant Stars

On a clear evening during the winter months, if you happen to be in the Northern Hemisphere and look up at the sky, you can see the constellation Orion (The Hunter). One star in this constellation, Rigel, flickers in a blue color and another star, Betelgeuse, has a reddish color, as shown in **Figure 6.5**. Which of these two stars is cooler, Betelgeuse or Rigel?

Strategy

We treat each star as a blackbody. Then according to Wien's law, its temperature is inversely proportional to the wavelength of its peak intensity. The wavelength $\lambda_{\text{max}}^{(\text{blue})}$ of blue light is shorter than the wavelength $\lambda_{\text{max}}^{(\text{red})}$ of red light. Even if we do not know the precise wavelengths, we can still set up a proportion.

Solution

Writing Wien's law for the blue star and for the red star, we have

$$\lambda_{\text{max}}^{(\text{red})} T_{(\text{red})} = 2.898 \times 10^{-3}\,\text{m} \cdot \text{K} = \lambda_{\text{max}}^{(\text{blue})} T_{(\text{blue})} \tag{6.2}$$

When simplified, **Equation 6.2** gives

$$T_{(\text{red})} = \frac{\lambda_{\text{max}}^{(\text{blue})}}{\lambda_{\text{max}}^{(\text{red})}} T_{(\text{blue})} < T_{(\text{blue})} \tag{6.3}$$

Therefore, Betelgeuse is cooler than Rigel.

Significance

Note that Wien's displacement law tells us that the higher the temperature of an emitting body, the shorter the wavelength of the radiation it emits. The qualitative analysis presented in this example is generally valid for any emitting body, whether it is a big object such as a star or a small object such as the glowing filament in an incandescent lightbulb.

6.1 Check Your Understanding The flame of a peach-scented candle has a yellowish color and the flame of a Bunsen's burner in a chemistry lab has a bluish color. Which flame has a higher temperature?

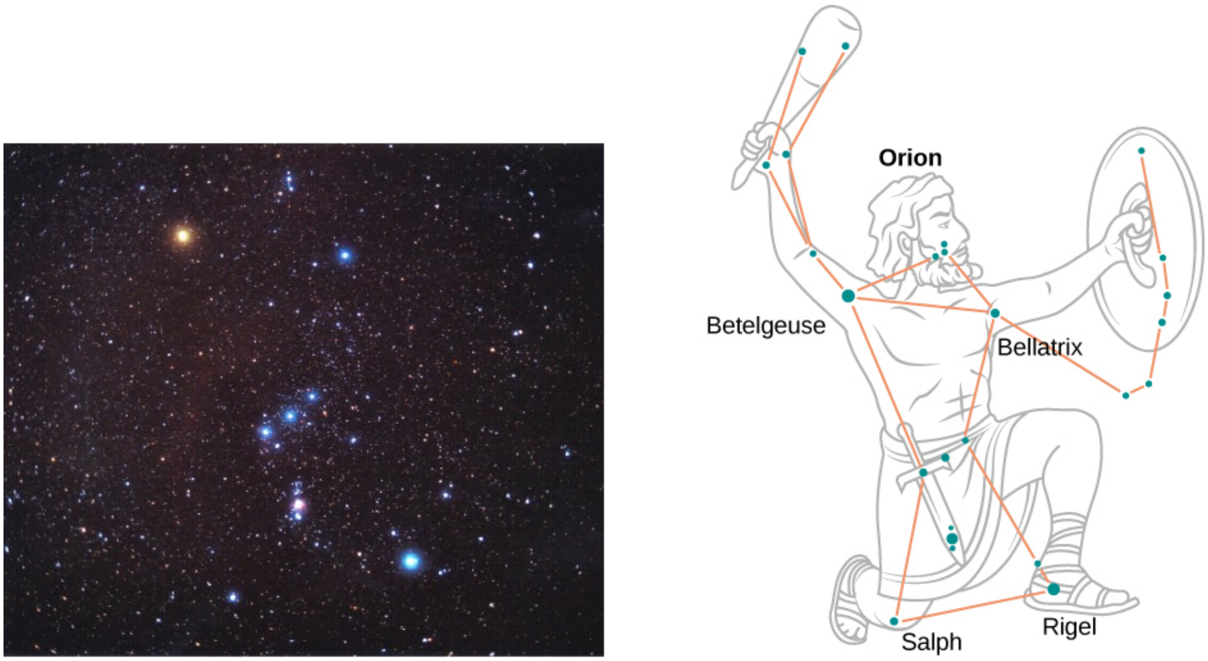


Figure 6.5 In the Orion constellation, the red star Betelgeuse, which usually takes on a yellowish tint, appears as the figure's right shoulder (in the upper left). The giant blue star on the bottom right is Rigel, which appears as the hunter's left foot. (credit left: modification of work by NASA c/o Matthew Spinelli)

The second experimental relation is Stefan's law**,** which concerns the total power of blackbody radiation emitted across the entire spectrum of wavelengths at a given temperature. In **Figure 6.3**, this total power is represented by the area under the blackbody radiation curve for a given T. As the temperature of a blackbody increases, the total emitted power also increases. Quantitatively, Stefan's law expresses this relation as

$$P(T) = \sigma A T^4 \tag{6.4}$$

where A is the surface area of a blackbody, T is its temperature (in kelvins), and σ is the **Stefan–Boltzmann constant**, $\sigma = 5.670 \times 10^{-8}\,\text{W/(m}^2 \cdot \text{K}^4\text{)}$. Stefan's law enables us to estimate how much energy a star is radiating by remotely measuring its temperature.

Example 6.2

Power Radiated by Stars

A star such as our Sun will eventually evolve to a "red giant" star and then to a "white dwarf" star. A typical white dwarf is approximately the size of Earth, and its surface temperature is about $2.5 \times 10^4\,\text{K}$. A typical red giant has a surface temperature of $3.0 \times 10^3\,\text{K}$ and a radius ~100,000 times larger than that of a white dwarf. What is the average radiated power per unit area and the total power radiated by each of these types of stars? How do they compare?

Strategy

If we treat the star as a blackbody, then according to Stefan's law, the total power that the star radiates is proportional to the fourth power of its temperature. To find the power radiated per unit area of the surface, we do not need to make any assumptions about the shape of the star because P/A depends only on temperature. However, to compute the total power, we need to make an assumption that the energy radiates through a spherical surface enclosing the star, so that the surface area is $A = 4\pi R^2$, where R is its radius.

Solution

A simple proportion based on Stefan's law gives

$$\frac{P_{\text{dwarf}}/A_{\text{dwarf}}}{P_{\text{giant}}/A_{\text{giant}}} = \frac{\sigma T_{\text{dwarf}}^4}{\sigma T_{\text{giant}}^4} = \left(\frac{T_{\text{dwarf}}}{T_{\text{giant}}}\right)^4 = \left(\frac{2.5 \times 10^4}{3.0 \times 10^3}\right)^4 = 4820 \qquad (6.5)$$

The power emitted per unit area by a white dwarf is about 5000 times that the power emitted by a red giant. Denoting this ratio by $a = 4.8 \times 10^3$, **Equation 6.5** gives

$$\frac{P_{\text{dwarf}}}{P_{\text{giant}}} = a\frac{A_{\text{dwarf}}}{A_{\text{giant}}} = a\frac{4\pi R_{\text{dwarf}}^2}{4\pi R_{\text{giant}}^2} = a\left(\frac{R_{\text{dwarf}}}{R_{\text{giant}}}\right)^2 = 4.8 \times 10^3 \left(\frac{R_{\text{dwarf}}}{10^5 R_{\text{dwarf}}}\right)^2 = 4.8 \times 10^{-7} \qquad (6.6)$$

We see that the total power emitted by a white dwarf is a tiny fraction of the total power emitted by a red giant. Despite its relatively lower temperature, the overall power radiated by a red giant far exceeds that of the white dwarf because the red giant has a much larger surface area. To estimate the absolute value of the emitted power per unit area, we again use Stefan's law. For the white dwarf, we obtain

$$\frac{P_{\text{dwarf}}}{A_{\text{dwarf}}} = \sigma T_{\text{dwarf}}^4 = 5.670 \times 10^{-8} \frac{\text{W}}{\text{m}^2 \cdot \text{K}^4}\left(2.5 \times 10^4\,\text{K}\right)^4 = 2.2 \times 10^{10}\,\text{W/m}^2 \qquad (6.7)$$

The analogous result for the red giant is obtained by scaling the result for a white dwarf:

$$\frac{P_{\text{giant}}}{A_{\text{giant}}} = \frac{2.2 \times 10^{10}}{4.82 \times 10^3}\frac{\text{W}}{\text{m}^2} = 4.56 \times 10^6 \frac{\text{W}}{\text{m}^2} \cong 4.6 \times 10^6 \frac{\text{W}}{\text{m}^2} \qquad (6.8)$$

Significance

To estimate the total power emitted by a white dwarf, in principle, we could use **Equation 6.7**. However, to find its surface area, we need to know the average radius, which is not given in this example. Therefore, the solution stops here. The same is also true for the red giant star.

6.2 Check Your Understanding An iron poker is being heated. As its temperature rises, the poker begins to glow—first dull red, then bright red, then orange, and then yellow. Use either the blackbody radiation curve or Wien's law to explain these changes in the color of the glow.

6.3 Check Your Understanding Suppose that two stars, α and β, radiate exactly the same total power. If the radius of star α is three times that of star β, what is the ratio of the surface temperatures of these stars? Which one is hotter?

The term "blackbody" was coined by Gustav R. Kirchhoff in 1862. The blackbody radiation curve was known experimentally, but its shape eluded physical explanation until the year 1900. The physical model of a blackbody at temperature T is that of the electromagnetic waves enclosed in a cavity (see **Figure 6.2**) and at thermodynamic equilibrium with the cavity walls. The waves can exchange energy with the walls. The objective here is to find the energy density distribution among various modes of vibration at various wavelengths (or frequencies). In other words, we want to know how much energy is carried by a single wavelength or a band of wavelengths. Once we know the energy distribution, we can use standard statistical methods (similar to those studied in a previous chapter) to obtain the blackbody radiation curve, Stefan's law, and Wien's displacement law. When the physical model is correct, the theoretical predictions should be the same as the experimental curves.

In a classical approach to the blackbody radiation problem, in which radiation is treated as waves (as you have studied in previous chapters), the modes of electromagnetic waves trapped in the cavity are in equilibrium and continually exchange their energies with the cavity walls. There is no physical reason why a wave should do otherwise: Any amount of energy can be exchanged, either by being transferred from the wave to the material in the wall or by being received by the wave from the material in the wall. This classical picture is the basis of the model developed by Lord Rayleigh and, independently, by Sir James Jeans. The result of this classical model for blackbody radiation curves is known as the *Rayleigh–Jeans law*.

However, as shown in **Figure 6.6**, the Rayleigh–Jeans law fails to correctly reproduce experimental results. In the limit of short wavelengths, the Rayleigh–Jeans law predicts infinite radiation intensity, which is inconsistent with the experimental results in which radiation intensity has finite values in the ultraviolet region of the spectrum. This divergence between the results of classical theory and experiments, which came to be called the *ultraviolet catastrophe*, shows how classical physics fails to explain the mechanism of blackbody radiation.

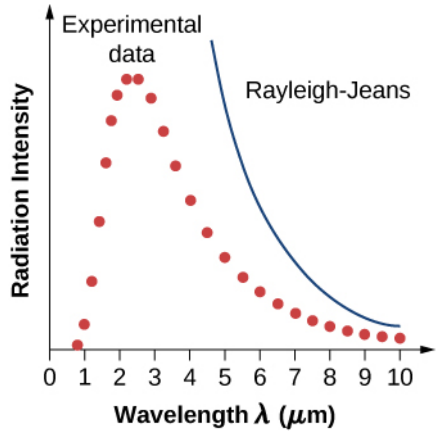


Figure 6.6 The ultraviolet catastrophe: The Rayleigh–Jeans law does not explain the observed blackbody emission spectrum.

The blackbody radiation problem was solved in 1900 by Max Planck. Planck used the same idea as the Rayleigh–Jeans model in the sense that he treated the electromagnetic waves between the walls inside the cavity classically, and assumed that the radiation is in equilibrium with the cavity walls. The innovative idea that Planck introduced in his model is the assumption that the cavity radiation originates from atomic oscillations inside the cavity walls, and that these oscillations can have only *discrete* values of energy. Therefore, the radiation trapped inside the cavity walls can exchange energy with the walls only in discrete amounts. Planck's hypothesis of discrete energy values, which he called *quanta*, assumes that the oscillators inside the cavity walls have **quantized energies**. This was a brand new idea that went beyond the classical physics of the nineteenth century because, as you learned in a previous chapter, in the classical picture, the energy of an oscillator can take on any continuous value. Planck assumed that the energy of an oscillator (E_n) can have only discrete, or quantized, values:

$$E_n = nhf, \quad \text{where } n = 1,\ 2,\ 3,\ ... \tag{6.9}$$

In **Equation 6.9**, f is the frequency of Planck's oscillator. The natural number n that enumerates these discrete energies is called a **quantum number**. The physical constant h is called *Planck's constant*:

$$h = 6.626 \times 10^{-34}\,\text{J} \cdot \text{s} = 4.136 \times 10^{-15}\,\text{eV} \cdot \text{s} \tag{6.10}$$

Each discrete energy value corresponds to a **quantum state of a Planck oscillator**. Quantum states are enumerated by quantum numbers. For example, when Planck's oscillator is in its first $n = 1$ quantum state, its energy is $E_1 = hf;$ when it is in the $n = 2$ quantum state, its energy is $E_2 = 2hf;$ when it is in the $n = 3$ quantum state, $E_3 = 3hf;$ and so on.

Note that **Equation 6.9** shows that there are infinitely many quantum states, which can be represented as a sequence {hf, $2hf$, $3hf$,…, $(n – 1)hf$, nhf, $(n + 1)hf$,…}. Each two consecutive quantum states in this sequence are separated by an energy jump, $\Delta E = hf.$ An oscillator in the wall can receive energy from the radiation in the cavity (absorption), or it can give away energy to the radiation in the cavity (emission). The absorption process sends the oscillator to a higher quantum state, and the emission process sends the oscillator to a lower quantum state. Whichever way this exchange of energy goes, the smallest amount of energy that can be exchanged is hf. There is no upper limit to how much energy can be exchanged, but whatever is exchanged must be an integer multiple of *hf*. If the energy packet does not have this exact amount, it is neither absorbed nor emitted at the wall of the blackbody.

Planck's Quantum Hypothesis

Planck's hypothesis of energy quanta states that the amount of energy emitted by the oscillator is carried by the quantum of radiation, ΔE :

$$\Delta E = hf$$

Recall that the frequency of electromagnetic radiation is related to its wavelength and to the speed of light by the fundamental relation $f\lambda = c$. This means that we can express **Equation 6.10** equivalently in terms of wavelength λ. When included in the computation of the energy density of a blackbody, Planck's hypothesis gives the following theoretical expression for the power intensity of emitted radiation per unit wavelength:

$$I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1} \tag{6.11}$$

where *c* is the speed of light in vacuum and k_B is Boltzmann's constant, $k_B = 1.380 \times 10^{-23}\,\text{J/K}$. The theoretical formula expressed in **Equation 6.11** is called *Planck's blackbody radiation law*. This law is in agreement with the experimental blackbody radiation curve (see **Figure 6.7**). In addition, Wien's displacement law and Stefan's law can both be derived from **Equation 6.11**. To derive Wien's displacement law, we use differential calculus to find the maximum of the radiation intensity curve $I(\lambda, T)$. To derive Stefan's law and find the value of the Stefan–Boltzmann constant, we use integral calculus and integrate $I(\lambda, T)$ to find the total power radiated by a blackbody at one temperature in the entire spectrum of wavelengths from $\lambda = 0$ to $\lambda = \infty$. This derivation is left as an exercise later in this chapter.

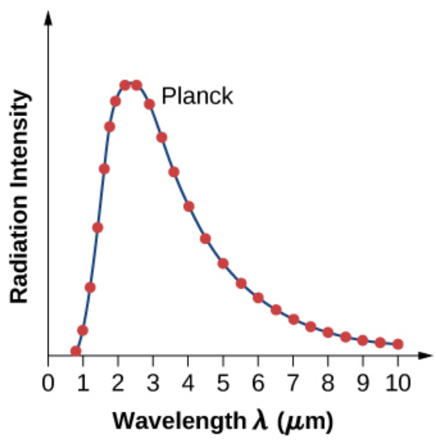


Figure 6.7 Planck's theoretical result (continuous curve) and the experimental blackbody radiation curve (dots).

Example 6.3

Planck's Quantum Oscillator

A quantum oscillator in the cavity wall in **Figure 6.2** is vibrating at a frequency of $5.0 \times 10^{14}\,\text{Hz}$. Calculate the spacing between its energy levels.

Strategy

Energy states of a quantum oscillator are given by **Equation 6.9**. The energy spacing ΔE is obtained by finding the energy difference between two adjacent quantum states for quantum numbers $n + 1$ and n.

Solution

We can substitute the given frequency and Planck's constant directly into the equation:

$$\Delta E = E_{n+1} - E_n = (n+1)hf - nhf = hf = (6.626 \times 10^{-34}\,\text{J} \cdot \text{s})(5.0 \times 10^{14}\,\text{Hz}) = 3.3 \times 10^{-19}\,\text{J}$$

Significance

Note that we do not specify what kind of material was used to build the cavity. Here, a quantum oscillator is a theoretical model of an atom or molecule of material in the wall.

6.4 Check Your Understanding A molecule is vibrating at a frequency of $5.0 \times 10^{14}\,\text{Hz}$. What is the smallest spacing between its vibrational energy levels?

Example 6.4

Quantum Theory Applied to a Classical Oscillator

A 1.0-kg mass oscillates at the end of a spring with a spring constant of 1000 N/m. The amplitude of these oscillations is 0.10 m. Use the concept of quantization to find the energy spacing for this classical oscillator. Is the energy quantization significant for macroscopic systems, such as this oscillator?

Strategy

We use **Equation 6.10** as though the system were a quantum oscillator, but with the frequency f of the mass vibrating on a spring. To evaluate whether or not quantization has a significant effect, we compare the quantum energy spacing with the macroscopic total energy of this classical oscillator.

Solution

For the spring constant, $k = 1.0 \times 10^3\,\text{N/m},$ the frequency f of the mass, $m = 1.0\text{kg},$ is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{1.0 \times 10^3\,\text{N/m}}{1.0\text{kg}}} \simeq 5.0\,\text{Hz}$$

The energy quantum that corresponds to this frequency is

$$\Delta E = hf = (6.626 \times 10^{-34}\,\text{J} \cdot \text{s})(5.0\text{Hz}) = 3.3 \times 10^{-33}\,\text{J}$$

When vibrations have amplitude $A = 0.10\text{m},$ the energy of oscillations is

$$E = \frac{1}{2}kA^2 = \frac{1}{2}(1000\text{N/m})(0.1\text{m})^2 = 5.0\text{J}$$

Significance

Thus, for a classical oscillator, we have $\Delta E/E \approx 10^{-34}$. We see that the separation of the energy levels is immeasurably small. Therefore, for all practical purposes, the energy of a classical oscillator takes on continuous values. This is why classical principles may be applied to macroscopic systems encountered in everyday life without loss of accuracy.

6.5 Check Your Understanding Would the result in **Example 6.4** be different if the mass were not 1.0 kg but a tiny mass of 1.0 μg, and the amplitude of vibrations were 0.10 μm?

When Planck first published his result, the hypothesis of energy quanta was not taken seriously by the physics community because it did not follow from any established physics theory at that time. It was perceived, even by Planck himself, as a useful mathematical trick that led to a good theoretical "fit" to the experimental curve. This perception was changed in 1905 when Einstein published his explanation of the photoelectric effect, in which he gave Planck's energy quantum a new meaning: that of a particle of light.

6.2 | Photoelectric Effect

Learning Objectives

By the end of this section you will be able to:

- Describe physical characteristics of the photoelectric effect
- Explain why the photoelectric effect cannot be explained by classical physics
- Describe how Einstein's idea of a particle of radiation explains the photoelectric effect

When a metal surface is exposed to a monochromatic electromagnetic wave of sufficiently short wavelength (or equivalently, above a threshold frequency), the incident radiation is absorbed and the exposed surface emits electrons. This phenomenon is known as the **photoelectric effect**. Electrons that are emitted in this process are called **photoelectrons**.

The experimental setup to study the photoelectric effect is shown schematically in Figure 6.8. The target material serves as the anode, which becomes the emitter of photoelectrons when it is illuminated by monochromatic radiation. We call this electrode the **photoelectrode**. Photoelectrons are collected at the cathode, which is kept at a lower potential with respect to the anode. The potential difference between the electrodes can be increased or decreased, or its polarity can be reversed. The electrodes are enclosed in an evacuated glass tube so that photoelectrons do not lose their kinetic energy on collisions with air molecules in the space between electrodes.

When the target material is not exposed to radiation, no current is registered in this circuit because the circuit is broken (note, there is a gap between the electrodes). But when the target material is connected to the negative terminal of a battery and exposed to radiation, a current is registered in this circuit; this current is called the **photocurrent**. Suppose that we now reverse the potential difference between the electrodes so that the target material now connects with the positive terminal of a battery, and then we slowly increase the voltage. The photocurrent gradually dies out and eventually stops flowing completely at some value of this reversed voltage. The potential difference at which the photocurrent stops flowing is called the **stopping potential**.

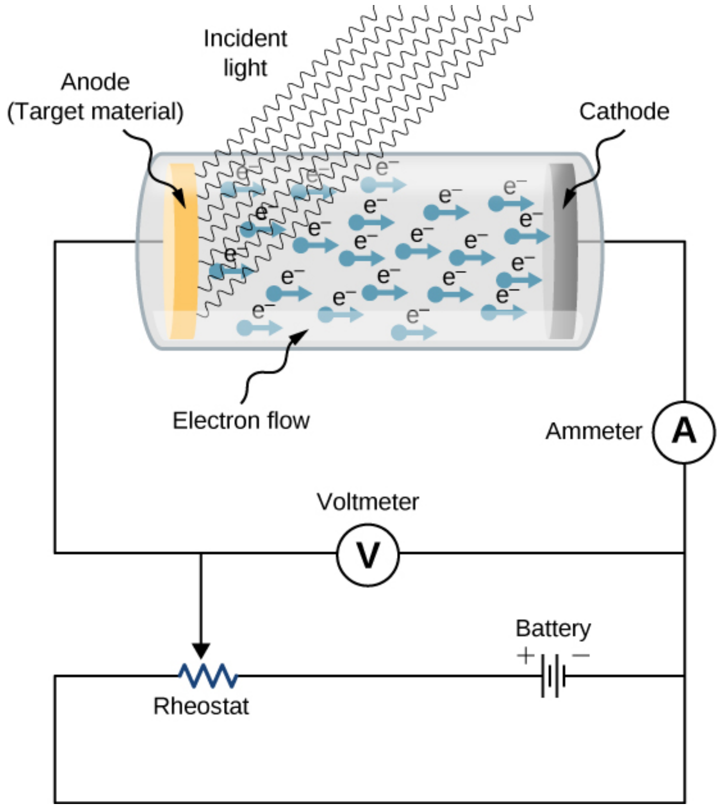


Figure 6.8 An experimental setup to study the photoelectric effect. The anode and cathode are enclosed in an evacuated glass tube. The voltmeter measures the electric potential difference between the electrodes, and the ammeter measures the photocurrent. The incident radiation is monochromatic.

Characteristics of the Photoelectric Effect

The photoelectric effect has three important characteristics that cannot be explained by classical physics: (1) the absence of a lag time, (2) the independence of the kinetic energy of photoelectrons on the intensity of incident radiation, and (3) the presence of a cut-off frequency. Let's examine each of these characteristics.

The absence of lag time

When radiation strikes the target material in the electrode, electrons are emitted almost instantaneously, even at very low intensities of incident radiation. This absence of lag time contradicts our understanding based on classical physics. Classical physics predicts that for low-energy radiation, it would take significant time before irradiated electrons could gain sufficient energy to leave the electrode surface; however, such an energy buildup is not observed.

The intensity of incident radiation and the kinetic energy of photoelectrons

Typical experimental curves are shown in **Figure 6.9**, in which the photocurrent is plotted versus the applied potential difference between the electrodes. For the positive potential difference, the current steadily grows until it reaches a plateau. Furthering the potential increase beyond this point does not increase the photocurrent at all. A higher intensity of radiation produces a higher value of photocurrent. For the negative potential difference, as the absolute value of the potential difference increases, the value of the photocurrent decreases and becomes zero at the stopping potential. For any intensity of incident radiation, whether the intensity is high or low, the value of the stopping potential always stays at one value.

To understand why this result is unusual from the point of view of classical physics, we first have to analyze the energy of photoelectrons. A photoelectron that leaves the surface has kinetic energy K. It gained this energy from the incident electromagnetic wave. In the space between the electrodes, a photoelectron moves in the electric potential and its energy changes by the amount $q\Delta V$, where ΔV is the potential difference and $q = -e$. Because no forces are present but electric force, by applying the work-energy theorem, we obtain the energy balance $\Delta K - e\Delta V = 0$ for the photoelectron, where ΔK is the change in the photoelectron's kinetic energy. When the stopping potential $-\Delta V_s$ is applied, the photoelectron loses its initial kinetic energy K_i and comes to rest. Thus, its energy balance becomes

$(0 - K_i) - e(-\Delta V_s) = 0,$ so that $K_i = e\Delta V_s$. In the presence of the stopping potential, the largest kinetic energy K_{max} that a photoelectron can have is its initial kinetic energy, which it has at the surface of the photoelectrode. Therefore, the largest kinetic energy of photoelectrons can be directly measured by measuring the stopping potential:

$$K_{max} = e\Delta V_s. \tag{6.12}$$

At this point we can see where the classical theory is at odds with the experimental results. In classical theory, the photoelectron absorbs electromagnetic energy in a continuous way; this means that when the incident radiation has a high intensity, the kinetic energy in **Equation 6.12** is expected to be high. Similarly, when the radiation has a low intensity, the kinetic energy is expected to be low. But the experiment shows that the maximum kinetic energy of photoelectrons is independent of the light intensity.

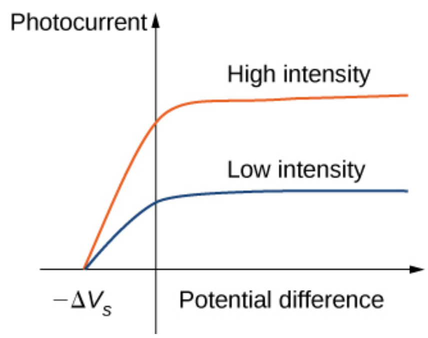


Figure 6.9 The detected photocurrent plotted versus the applied potential difference shows that for any intensity of incident radiation, whether the intensity is high (upper curve) or low (lower curve), the value of the stopping potential is always the same.

The presence of a cut-off frequency

For any metal surface, there is a minimum frequency of incident radiation below which photocurrent does not occur. The value of this **cut-off frequency** for the photoelectric effect is a physical property of the metal: Different materials have different values of cut-off frequency. Experimental data show a typical linear trend (see **Figure 6.10**). The kinetic energy of photoelectrons at the surface grows linearly with the increasing frequency of incident radiation. Measurements for all metal surfaces give linear plots with one slope. None of these observed phenomena is in accord with the classical understanding of nature. According to the classical description, the kinetic energy of photoelectrons should not depend on the frequency of incident radiation at all, and there should be no cut-off frequency. Instead, in the classical picture, electrons receive energy from the incident electromagnetic wave in a continuous way, and the amount of energy they receive depends only on the intensity of the incident light and nothing else. So in the classical understanding, as long as the light is shining, the photoelectric effect is expected to continue.

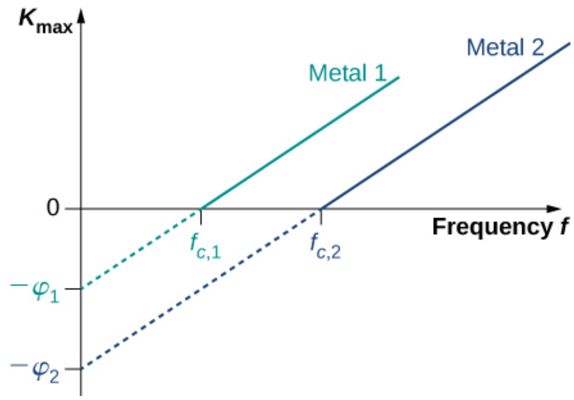


Figure 6.10 Kinetic energy of photoelectrons at the surface versus the frequency of incident radiation. The photoelectric effect can only occur above the cut-off frequency f_c.

Measurements for all metal surfaces give linear plots with one slope. Each metal surface has its own cut-off frequency.

The Work Function

The photoelectric effect was explained in 1905 by A. Einstein. Einstein reasoned that if Planck's hypothesis about energy quanta was correct for describing the energy exchange between electromagnetic radiation and cavity walls, it should also work to describe energy absorption from electromagnetic radiation by the surface of a photoelectrode. He postulated that an electromagnetic wave carries its energy in discrete packets. Einstein's postulate goes beyond Planck's hypothesis because it states that the light itself consists of energy quanta. In other words, it states that electromagnetic waves are quantized.

In Einstein's approach, a beam of monochromatic light of frequency f is made of photons. A **photon** is a particle of light. Each photon moves at the speed of light and carries an energy quantum E_f. A photon's energy depends only on its frequency f. Explicitly, the **energy of a photon** is

$$E_f = hf \tag{6.13}$$

where h is Planck's constant. In the photoelectric effect, photons arrive at the metal surface and each photon gives away *all* of its energy to only *one* electron on the metal surface. This transfer of energy from photon to electron is of the "all or nothing" type, and there are no fractional transfers in which a photon would lose only part of its energy and survive. The essence of a **quantum phenomenon** is either a photon transfers its entire energy and ceases to exist or there is no transfer at all. This is in contrast with the classical picture, where fractional energy transfers are permitted. Having this quantum understanding, the energy balance for an electron on the surface that receives the energy E_f from a photon is

$$E_f = K_{\text{max}} + \phi$$

where K_{max} is the kinetic energy, given by **Equation 6.12**, that an electron has at the very instant it gets detached from the surface. In this energy balance equation, ϕ is the energy needed to detach a photoelectron from the surface. This energy ϕ is called the **work function** of the metal. Each metal has its characteristic work function, as illustrated in **Table 6.1**. To obtain the kinetic energy of photoelectrons at the surface, we simply invert the energy balance equation and use **Equation 6.13** to express the energy of the absorbed photon. This gives us the expression for the kinetic energy of photoelectrons, which explicitly depends on the frequency of incident radiation:

$$K_{\text{max}} = hf - \phi. \tag{6.14}$$

This equation has a simple mathematical form but its physics is profound. We can now elaborate on the physical meaning behind **Equation 6.14**.

Typical Values of the Work Function for Some Common Metals

Metal	ϕ (eV)
Na	2.46
Al	4.08
Pb	4.14
Zn	4.31
Fe	4.50
Cu	4.70
Ag	4.73
Pt	6.35

Table 6.1

In Einstein’s interpretation, interactions take place between individual electrons and individual photons. The absence of a lag time means that these one-on-one interactions occur instantaneously. This interaction time cannot be increased by lowering the light intensity. The light intensity corresponds to the number of photons arriving at the metal surface per unit time. Even at very low light intensities, the photoelectric effect still occurs because the interaction is between one electron and one photon. As long as there is at least one photon with enough energy to transfer it to a bound electron, a photoelectron will appear on the surface of the photoelectrode.

The existence of the cut-off frequency f_c for the photoelectric effect follows from **Equation 6.14** because the kinetic energy K_{max} of the photoelectron can take only positive values. This means that there must be some threshold frequency for which the kinetic energy is zero, $0 = hf_c - \phi$. In this way, we obtain the explicit formula for cut-off frequency:

$$f_c = \frac{\phi}{h}. \tag{6.15}$$

Cut-off frequency depends only on the work function of the metal and is in direct proportion to it. When the work function is large (when electrons are bound fast to the metal surface), the energy of the threshold photon must be large to produce a photoelectron, and then the corresponding threshold frequency is large. Photons with frequencies larger than the threshold frequency f_c always produce photoelectrons because they have $K_{\text{max}} > 0$. Photons with frequencies smaller than f_c do not have enough energy to produce photoelectrons. Therefore, when incident radiation has a frequency below the cut-off frequency, the photoelectric effect is not observed. Because frequency f and wavelength λ of electromagnetic waves are related by the fundamental relation $\lambda f = c$ (where c is the speed of light in vacuum), the cut-off frequency has its corresponding **cut-off wavelength** λ_c :

$$\lambda_c = \frac{c}{f_c} = \frac{c}{\phi/h} = \frac{hc}{\phi}. \tag{6.16}$$

In this equation, $hc = 1240\,\text{eV}\cdot\text{nm}$. Our observations can be restated in the following equivalent way: When the incident radiation has wavelengths longer than the cut-off wavelength, the photoelectric effect does not occur.

Example 6.5

Photoelectric Effect for Silver

Radiation with wavelength 300 nm is incident on a silver surface. Will photoelectrons be observed?

Strategy

Photoelectrons can be ejected from the metal surface only when the incident radiation has a shorter wavelength than the cut-off wavelength. The work function of silver is $\phi = 4.73 \text{ eV}$ (**Table 6.1**). To make the estimate, we use **Equation 6.16**.

Solution

The threshold wavelength for observing the photoelectric effect in silver is

$$\lambda_c = \frac{hc}{\phi} = \frac{1240 \text{ eV} \cdot \text{nm}}{4.73 \text{ eV}} = 262 \text{ nm}.$$

The incident radiation has wavelength 300 nm, which is longer than the cut-off wavelength; therefore, photoelectrons are not observed.

Significance

If the photoelectrode were made of sodium instead of silver, the cut-off wavelength would be 504 nm and photoelectrons would be observed.

Equation 6.14 in Einstein's model tells us that the maximum kinetic energy of photoelectrons is a linear function of the frequency of incident radiation, which is illustrated in **Figure 6.10**. For any metal, the slope of this plot has a value of Planck's constant. The intercept with the K_{max} -axis gives us a value of the work function that is characteristic for the metal. On the other hand, K_{max} can be directly measured in the experiment by measuring the value of the stopping potential ΔV_s (see **Equation 6.12**) at which the photocurrent stops. These direct measurements allow us to determine experimentally the value of Planck's constant, as well as work functions of materials.

Einstein's model also gives a straightforward explanation for the photocurrent values shown in **Figure 6.9**. For example, doubling the intensity of radiation translates to doubling the number of photons that strike the surface per unit time. The larger the number of photons, the larger is the number of photoelectrons, which leads to a larger photocurrent in the circuit. This is how radiation intensity affects the photocurrent. The photocurrent must reach a plateau at some value of potential difference because, in unit time, the number of photoelectrons is equal to the number of incident photons and the number of incident photons does not depend on the applied potential difference at all, but only on the intensity of incident radiation. The stopping potential does not change with the radiation intensity because the kinetic energy of photoelectrons (see **Equation 6.14**) does not depend on the radiation intensity.

Example 6.6

Work Function and Cut-Off Frequency

When a 180-nm light is used in an experiment with an unknown metal, the measured photocurrent drops to zero at potential – 0.80 V. Determine the work function of the metal and its cut-off frequency for the photoelectric effect.

Strategy

To find the cut-off frequency $f_c,$ we use **Equation 6.15**, but first we must find the work function ϕ. To find $\phi,$ we use **Equation 6.12** and **Equation 6.14**. Photocurrent drops to zero at the stopping value of potential, so we identify $\Delta V_s = 0.8\text{V}$.

Solution

We use **Equation 6.12** to find the kinetic energy of the photoelectrons:

$$K_{\text{max}} = e\Delta V_s = e(0.80\text{V}) = 0.80 \text{ eV}.$$

Now we solve **Equation 6.14** for ϕ :

$$\phi = hf - K_{\text{max}} = \frac{hc}{\lambda} - K_{\text{max}} = \frac{1240 \text{ eV} \cdot \text{nm}}{180 \text{ nm}} - 0.80 \text{ eV} = 6.09 \text{ eV}.$$

Finally, we use **Equation 6.15** to find the cut-off frequency:

$$f_c = \frac{\phi}{h} = \frac{6.09\ \text{eV}}{4.136 \times 10^{-15}\ \text{eV} \cdot \text{s}} = 1.47 \times 10^{-15}\ \text{Hz}.$$

Significance

In calculations like the one shown in this example, it is convenient to use Planck's constant in the units of $\text{eV} \cdot \text{s}$ and express all energies in eV instead of joules.

Example 6.7

The Photon Energy and Kinetic Energy of Photoelectrons

A 430-nm violet light is incident on a calcium photoelectrode with a work function of 2.71 eV.

Find the energy of the incident photons and the maximum kinetic energy of ejected electrons.

Strategy

The energy of the incident photon is $E_f = hf = hc/\lambda,$ where we use $f\lambda = c.$ To obtain the maximum energy of the ejected electrons, we use **Equation 6.16**.

Solution

$$E_f = \frac{hc}{\lambda} = \frac{1240\ \text{eV} \cdot \text{nm}}{430\ \text{nm}} = 2.88\ \text{eV}, \ \ K_{\text{max}} = E_f - \phi = 2.88\ \text{eV} - 2.71\ \text{eV} = 0.17\ \text{eV}$$

Significance

In this experimental setup, photoelectrons stop flowing at the stopping potential of 0.17 V.

6.6 Check Your Understanding A yellow 589-nm light is incident on a surface whose work function is 1.20 eV. What is the stopping potential? What is the cut-off wavelength?

6.7 Check Your Understanding Cut-off frequency for the photoelectric effect in some materials is $8.0 \times 10^{13}\ \text{Hz}.$ When the incident light has a frequency of $1.2 \times 10^{14}\ \text{Hz},$ the stopping potential is measured as – 0.16 V. Estimate a value of Planck's constant from these data (in units $\text{J} \cdot \text{s}$ and $\text{eV} \cdot \text{s}$) and determine the percentage error of your estimation.

6.3 | The Compton Effect

Learning Objectives

By the end of this section, you will be able to:

- Describe Compton's experiment
- Explain the Compton wavelength shift
- Describe how experiments with X-rays confirm the particle nature of radiation

Two of Einstein's influential ideas introduced in 1905 were the theory of special relativity and the concept of a light quantum, which we now call a photon. Beyond 1905, Einstein went further to suggest that freely propagating electromagnetic waves consisted of photons that are particles of light in the same sense that electrons or other massive particles are particles of matter. A beam of monochromatic light of wavelength λ (or equivalently, of frequency *f*) can be seen either as a classical wave or as a collection of photons that travel in a vacuum with one speed, *c* (the speed of light), and all carrying the same energy, $E_f = hf$. This idea proved useful for explaining the interactions of light with particles of matter.

Momentum of a Photon

Unlike a particle of matter that is characterized by its rest mass m_0, a photon is massless. In a vacuum, unlike a particle of matter that may vary its speed but cannot reach the speed of light, a photon travels at only one speed, which is exactly the speed of light. From the point of view of Newtonian classical mechanics, these two characteristics imply that a photon should not exist at all. For example, how can we find the linear momentum or kinetic energy of a body whose mass is zero? This apparent paradox vanishes if we describe a photon as a relativistic particle. According to the theory of special relativity, any particle in nature obeys the relativistic energy equation

$$E^2 = p^2c^2 + m_0^2c^4. \quad (6.17)$$

This relation can also be applied to a photon. In **Equation 6.17**, E is the total energy of a particle, p is its linear momentum, and m_0 is its rest mass. For a photon, we simply set $m_0 = 0$ in this equation. This leads to the expression for the momentum p_f of a photon

$$p_f = \frac{E_f}{c}. \quad (6.18)$$

Here the photon's energy E_f is the same as that of a light quantum of frequency f, which we introduced to explain the photoelectric effect:

$$E_f = hf = \frac{hc}{\lambda}. \quad (6.19)$$

The wave relation that connects frequency f with wavelength λ and speed c also holds for photons:

$$\lambda f = c \quad (6.20)$$

Therefore, a photon can be equivalently characterized by either its energy and wavelength, or its frequency and momentum. **Equation 6.19** and **Equation 6.20** can be combined into the explicit relation between a photon's momentum and its wavelength:

$$p_f = \frac{h}{\lambda}. \quad (6.21)$$

Notice that this equation gives us only the magnitude of the photon's momentum and contains no information about the direction in which the photon is moving. To include the direction, it is customary to write the photon's momentum as a vector:

$$\vec{\mathbf{p}}_f = \hbar \vec{\mathbf{k}}. \quad (6.22)$$

In **Equation 6.22**, $\hbar = h/2\pi$ is the **reduced Planck's constant** (pronounced "h-bar"), which is just Planck's constant divided by the factor 2π. Vector $\vec{\mathbf{k}}$ is called the "wave vector" or propagation vector (the direction in which a photon is moving). The **propagation vector** shows the direction of the photon's linear momentum vector. The magnitude of the wave vector is $k = |\vec{\mathbf{k}}| = 2\pi/\lambda$ and is called the **wave number**. Notice that this equation does not introduce any new physics. We can verify that the magnitude of the vector in **Equation 6.22** is the same as that given by **Equation 6.18**.

The Compton Effect

The **Compton effect** is the term used for an unusual result observed when X-rays are scattered on some materials. By classical theory, when an electromagnetic wave is scattered off atoms, the wavelength of the scattered radiation is expected to be the same as the wavelength of the incident radiation. Contrary to this prediction of classical physics, observations show that when X-rays are scattered off some materials, such as graphite, the scattered X-rays have different wavelengths from the wavelength of the incident X-rays. This classically unexplainable phenomenon was studied experimentally by Arthur H. Compton and his collaborators, and Compton gave its explanation in 1923.

To explain the shift in wavelengths measured in the experiment, Compton used Einstein's idea of light as a particle. The Compton effect has a very important place in the history of physics because it shows that electromagnetic radiation cannot be explained as a purely wave phenomenon. The explanation of the Compton effect gave a convincing argument to the physics community that electromagnetic waves can indeed behave like a stream of photons, which placed the concept of a photon on firm ground.

The schematics of Compton's experimental setup are shown in **Figure 6.11**. The idea of the experiment is straightforward: Monochromatic X-rays with wavelength λ are incident on a sample of graphite (the "target"), where they interact with atoms inside the sample; they later emerge as scattered X-rays with wavelength λ'. A detector placed behind the target can measure the intensity of radiation scattered in any direction θ with respect to the direction of the incident X-ray beam. This **scattering angle**, θ, is the angle between the direction of the scattered beam and the direction of the incident beam. In this experiment, we know the intensity and the wavelength λ of the incoming (incident) beam; and for a given scattering angle θ, we measure the intensity and the wavelength λ' of the outgoing (scattered) beam. Typical results of these measurements are shown in **Figure 6.12**, where the *x*-axis is the wavelength of the scattered X-rays and the *y*-axis is the intensity of the scattered X-rays, measured for different scattering angles (indicated on the graphs). For all scattering angles (except for $\theta = 0°$), we measure two intensity peaks. One peak is located at the wavelength λ, which is the wavelength of the incident beam. The other peak is located at some other wavelength, λ'. The two peaks are separated by $\Delta\lambda$, which depends on the scattering angle θ of the outgoing beam (in the direction of observation). The separation $\Delta\lambda$ is called the **Compton shift**.

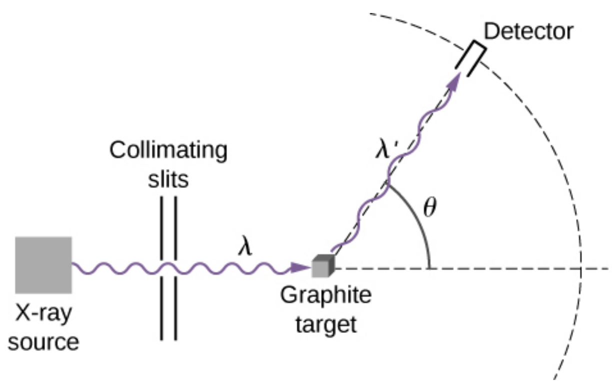


Figure 6.11 Experimental setup for studying Compton scattering.

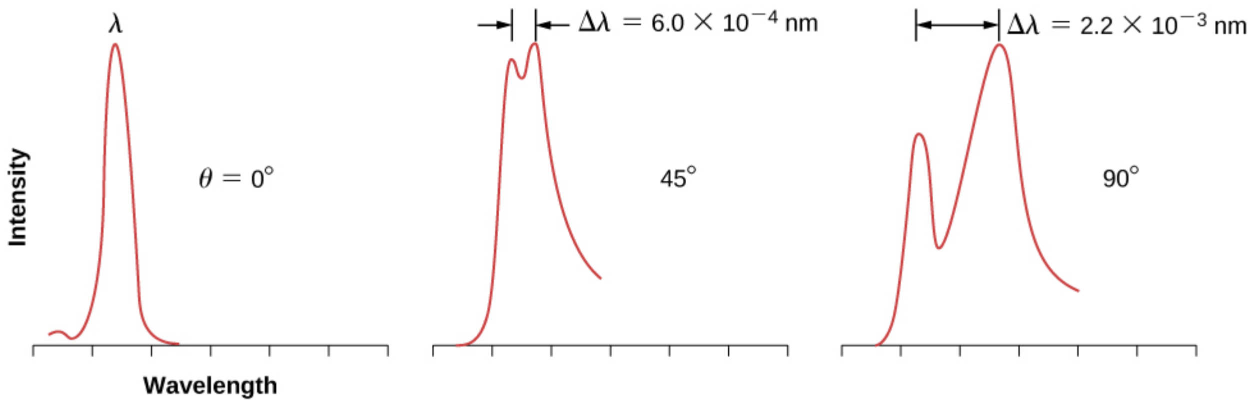


Figure 6.12 Experimental data show the Compton effect for X-rays scattering off graphite at various angles: The intensity of the scattered beam has two peaks. One peak appears at the wavelength λ of the incident radiation and the second peak appears at wavelength λ'. The separation $\Delta\lambda$ between the peaks depends on the scattering angle θ, which is the angular position of the detector in **Figure 6.11**. The experimental data in this figure are plotted in arbitrary units so that the height of the profile reflects the intensity of the scattered beam above background noise.

Compton Shift

As given by Compton, the explanation of the Compton shift is that in the target material, graphite, valence electrons are loosely bound in the atoms and behave like free electrons. Compton assumed that the incident X-ray radiation is a stream of photons. An incoming photon in this stream collides with a valence electron in the graphite target. In the course of this collision, the incoming photon transfers some part of its energy and momentum to the target electron and leaves the scene as a scattered photon. This model explains in qualitative terms why the scattered radiation has a longer wavelength than the incident radiation. Put simply, a photon that has lost some of its energy emerges as a photon with a lower frequency, or equivalently, with a longer wavelength. To show that his model was correct, Compton used it to derive the expression for the Compton shift. In his derivation, he assumed that both photon and electron are relativistic particles and that the collision obeys two commonsense principles: (1) the conservation of linear momentum and (2) the conservation of total relativistic energy.

In the following derivation of the Compton shift, E_f and $\vec{\mathbf{p}}_f$ denote the energy and momentum, respectively, of an incident photon with frequency *f*. The photon collides with a relativistic electron at rest, which means that immediately before the collision, the electron's energy is entirely its rest mass energy, m_0c^2. Immediately after the collision, the electron has energy *E* and momentum $\vec{\mathbf{p}}$, both of which satisfy **Equation 6.19**. Immediately after the collision, the outgoing photon has energy $\tilde{E}_f$, momentum $\vec{\tilde{\mathbf{p}}}_f$, and frequency f'. The direction of the incident photon is horizontal from left to right, and the direction of the outgoing photon is at the angle θ, as illustrated in **Figure 6.11**. The scattering angle θ is the angle between the momentum vectors $\vec{\mathbf{p}}_f$ and $\vec{\tilde{\mathbf{p}}}_f$, and we can write their scalar product:

$$\vec{\mathbf{p}}_f \cdot \vec{\tilde{\mathbf{p}}}_f = p_f \tilde{p}_f \cos\theta. \quad (6.23)$$

Following Compton's argument, we assume that the colliding photon and electron form an isolated system. This assumption is valid for weakly bound electrons that, to a good approximation, can be treated as free particles. Our first equation is the conservation of energy for the photon-electron system:

$$E_f + m_0c^2 = \tilde{E}_f + E. \quad (6.24)$$

The left side of this equation is the energy of the system at the instant immediately before the collision, and the right side of the equation is the energy of the system at the instant immediately after the collision. Our second equation is the conservation of linear momentum for the photon–electron system where the electron is at rest at the instant immediately before the collision:

$$\vec{\mathbf{p}}_f = \vec{\tilde{\mathbf{p}}}_f + \vec{\mathbf{p}}. \quad (6.25)$$

The left side of this equation is the momentum of the system right before the collision, and the right side of the equation is the momentum of the system right after collision. The entire physics of Compton scattering is contained in these three preceding equations—the remaining part is algebra. At this point, we could jump to the concluding formula for the Compton shift, but it is beneficial to highlight the main algebraic steps that lead to Compton's formula, which we give here as follows.

We start with rearranging the terms in **Equation 6.24** and squaring it:

$$\left[\left(E_f - \tilde{E}_f\right) + m_0 c^2\right]^2 = E^2.$$

In the next step, we substitute **Equation 6.19** for E^2, simplify, and divide both sides by c^2 to obtain

$$\left(E_f/c - \tilde{E}_f/c\right)^2 + 2m_0 c\left(E_f/c - \tilde{E}_f/c\right) = p^2.$$

Now we can use **Equation 6.21** to express this form of the energy equation in terms of momenta. The result is

$$\left(p_f - \tilde{p}_f\right)^2 + 2m_0 c\left(p_f - \tilde{p}_f\right) = p^2. \quad (6.26)$$

To eliminate p^2, we turn to the momentum equation **Equation 6.25**, rearrange its terms, and square it to obtain

$$\left(\vec{\mathbf{p}}_f - \vec{\tilde{\mathbf{p}}}_f\right)^2 = p^2 \text{ and } \left(\vec{\mathbf{p}}_f - \vec{\tilde{\mathbf{p}}}_f\right)^2 = p_f^2 + \tilde{p}_f^2 - 2\,\vec{\mathbf{p}}_f \cdot \vec{\tilde{\mathbf{p}}}_f.$$

The product of the momentum vectors is given by **Equation 6.23**. When we substitute this result for p^2 in **Equation 6.26**, we obtain the energy equation that contains the scattering angle θ :

$$\left(p_f - \tilde{p}_f\right)^2 + 2m_0 c\left(p_f - \tilde{p}_f\right) = p_f^2 + \tilde{p}_f^2 - 2p_f \tilde{p}_f \cos\theta.$$

With further algebra, this result can be simplified to

$$\frac{1}{\tilde{p}_f} - \frac{1}{p_f} = \frac{1}{m_0 c}(1 - \cos\theta). \quad (6.27)$$

Now recall **Equation 6.21** and write: $1/\tilde{p}_f = \lambda'/h$ and $1/p_f = \lambda/h$. When these relations are substituted into **Equation 6.27**, we obtain the relation for the Compton shift:

$$\lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos\theta). \quad (6.28)$$

The factor $h/m_0 c$ is called the **Compton wavelength** of the electron:

$$\lambda_c = \frac{h}{m_0 c} = 0.00243 \text{ nm} = 2.43 \text{ pm}. \quad (6.29)$$

Denoting the shift as $\Delta\lambda = \lambda' - \lambda$, the concluding result can be rewritten as

$$\Delta\lambda = \lambda_c(1 - \cos\theta). \quad (6.30)$$

This formula for the Compton shift describes outstandingly well the experimental results shown in **Figure 6.12**. Scattering data measured for molybdenum, graphite, calcite, and many other target materials are in accord with this theoretical result. The nonshifted peak shown in **Figure 6.12** is due to photon collisions with tightly bound inner electrons in the target material. Photons that collide with the inner electrons of the target atoms in fact collide with the entire atom. In this extreme case, the rest mass in **Equation 6.29** must be changed to the rest mass of the atom. This type of shift is four orders of magnitude smaller than the shift caused by collisions with electrons and is so small that it can be neglected.

Compton scattering is an example of **inelastic scattering**, in which the scattered radiation has a longer wavelength than the wavelength of the incident radiation. In today's usage, the term "Compton scattering" is used for the inelastic scattering of photons by free, charged particles. In Compton scattering, treating photons as particles with momenta that can be transferred to charged particles provides the theoretical background to explain the wavelength shifts measured in experiments; this is the evidence that radiation consists of photons.

Example 6.8

Compton Scattering

An incident 71-pm X-ray is incident on a calcite target. Find the wavelength of the X-ray scattered at a $30°$ angle. What is the largest shift that can be expected in this experiment?

Strategy

To find the wavelength of the scattered X-ray, first we must find the Compton shift for the given scattering angle, $\theta = 30°$. We use **Equation 6.30**. Then we add this shift to the incident wavelength to obtain the scattered wavelength. The largest Compton shift occurs at the angle θ when $1 - \cos\theta$ has the largest value, which is for the angle $\theta = 180°$.

Solution

The shift at $\theta = 30°$ is

$$\Delta\lambda = \lambda_c(1 - \cos 30°) = 0.134\lambda_c = (0.134)(2.43)\text{ pm} = 0.325\text{ pm}.$$

This gives the scattered wavelength:

$$\lambda' = \lambda + \Delta\lambda = (71 + 0.325)\text{ pm} = 71.325\text{ pm}.$$

The largest shift is

$$(\Delta\lambda)_{\text{max}} = \lambda_c(1 - \cos 180^0) = 2(2.43\text{ pm}) = 4.86\text{ pm}.$$

Significance

The largest shift in wavelength is detected for the backscattered radiation; however, most of the photons from the incident beam pass through the target and only a small fraction of photons gets backscattered (typically, less than 5%). Therefore, these measurements require highly sensitive detectors.

6.8 Check Your Understanding An incident 71-pm X-ray is incident on a calcite target. Find the wavelength of the X-ray scattered at a $60°$ angle. What is the smallest shift that can be expected in this experiment?

6.4 | Bohr's Model of the Hydrogen Atom

Learning Objectives

By the end of this section, you will be able to:

- Explain the difference between the absorption spectrum and the emission spectrum of radiation emitted by atoms
- Describe the Rutherford gold foil experiment and the discovery of the atomic nucleus
- Explain the atomic structure of hydrogen
- Describe the postulates of the early quantum theory for the hydrogen atom
- Summarize how Bohr's quantum model of the hydrogen atom explains the radiation spectrum of atomic hydrogen

Historically, Bohr's model of the hydrogen atom is the very first model of atomic structure that correctly explained the radiation spectra of atomic hydrogen. The model has a special place in the history of physics because it introduced an early quantum theory, which brought about new developments in scientific thought and later culminated in the development of quantum mechanics. To understand the specifics of Bohr's model, we must first review the nineteenth-century discoveries that prompted its formulation.

When we use a prism to analyze white light coming from the sun, several dark lines in the solar spectrum are observed (**Figure 6.13**). Solar absorption lines are called **Fraunhofer lines** after Joseph von Fraunhofer, who accurately measured their wavelengths. During 1854–1861, Gustav Kirchhoff and Robert Bunsen discovered that for the various chemical elements, the line **emission spectrum** of an element exactly matches its line **absorption spectrum**. The difference between the absorption spectrum and the emission spectrum is explained in **Figure 6.14**. An absorption spectrum is observed when light passes through a gas. This spectrum appears as black lines that occur only at certain wavelengths on the background of the continuous spectrum of white light (**Figure 6.13**). The missing wavelengths tell us which wavelengths of the radiation are absorbed by the gas. The emission spectrum is observed when light is emitted by a gas. This spectrum is seen as colorful lines on the black background (see **Figure 6.15** and **Figure 6.16**). Positions of the emission lines tell us which wavelengths of the radiation are emitted by the gas. Each chemical element has its own characteristic emission spectrum. For each element, the positions of its emission lines are exactly the same as the positions of its absorption lines. This means that atoms of a specific element absorb radiation only at specific wavelengths and radiation that does not have these wavelengths is not absorbed by the element at all. This also means that the radiation emitted by atoms of each element has exactly the same wavelengths as the radiation they absorb.

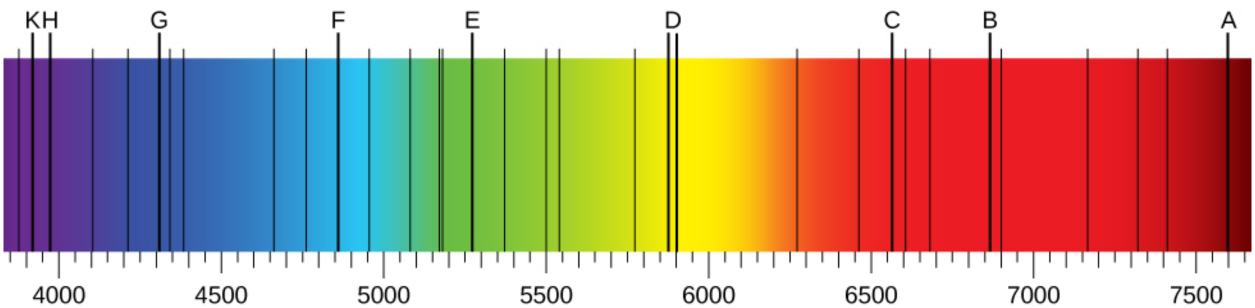


Figure 6.13 In the solar emission spectrum in the visible range from 380 nm to 710 nm, Fraunhofer lines are observed as vertical black lines at specific spectral positions in the continuous spectrum. Highly sensitive modern instruments observe thousands of such lines.

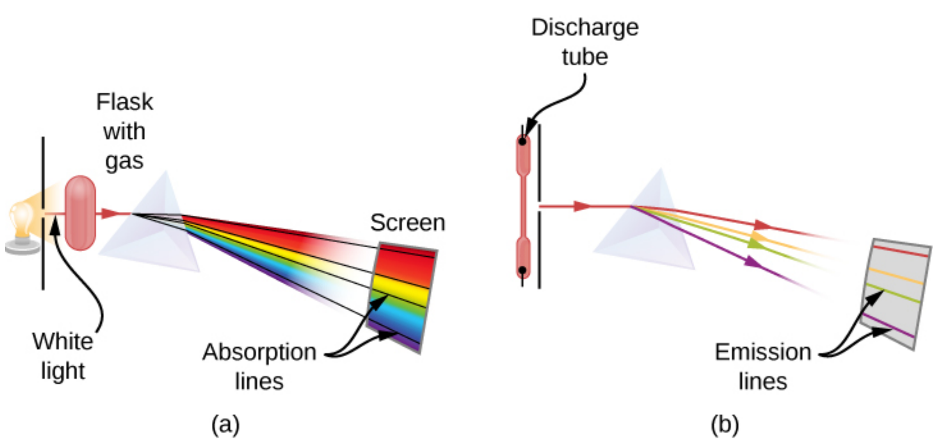


Figure 6.14 Observation of line spectra: (a) setup to observe absorption lines; (b) setup to observe emission lines. (a) White light passes through a cold gas that is contained in a glass flask. A prism is used to separate wavelengths of the passed light. In the spectrum of the passed light, some wavelengths are missing, which are seen as black absorption lines in the continuous spectrum on the viewing screen. (b) A gas is contained in a glass discharge tube that has electrodes at its ends. At a high potential difference between the electrodes, the gas glows and the light emitted from the gas passes through the prism that separates its wavelengths. In the spectrum of the emitted light, only specific wavelengths are present, which are seen as colorful emission lines on the screen.

Figure 6.15 The emission spectrum of atomic hydrogen: The spectral positions of emission lines are characteristic for hydrogen atoms. (credit: "Merikanto"/Wikimedia Commons)

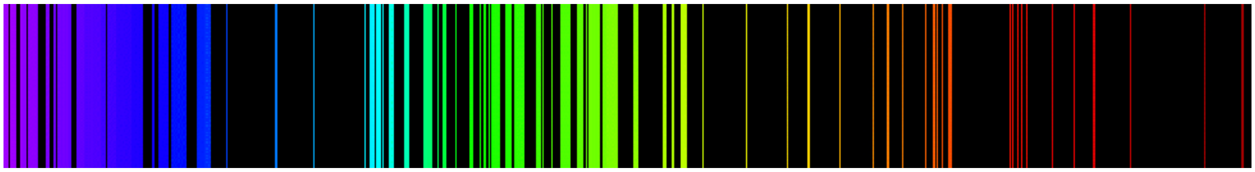


Figure 6.16 The emission spectrum of atomic iron: The spectral positions of emission lines are characteristic for iron atoms.

Emission spectra of the elements have complex structures; they become even more complex for elements with higher atomic numbers. The simplest spectrum, shown in **Figure 6.15**, belongs to the hydrogen atom. Only four lines are visible to the human eye. As you read from right to left in **Figure 6.15**, these lines are: red (656 nm), called the H-α line; aqua (486 nm), blue (434 nm), and violet (410 nm). The lines with wavelengths shorter than 400 nm appear in the ultraviolet part of the spectrum (**Figure 6.15**, far left) and are invisible to the human eye. There are infinitely many invisible spectral lines in the series for hydrogen.

An empirical formula to describe the positions (wavelengths) λ of the hydrogen emission lines in this series was discovered in 1885 by Johann Balmer. It is known as the **Balmer formula**:

$$\frac{1}{\lambda} = R_{\text{H}}\left(\frac{1}{2^2} - \frac{1}{n^2}\right). \tag{6.31}$$

The constant $R_{\text{H}} = 1.09737 \times 10^7\,\text{m}^{-1}$ is called the **Rydberg constant for hydrogen**. In **Equation 6.31**, the positive integer *n* takes on values $n = 3,\ 4,\ 5,\ 6$ for the four visible lines in this series. The series of emission lines given by the Balmer formula is called the **Balmer series** for hydrogen. Other emission lines of hydrogen that were discovered in the twentieth century are described by the **Rydberg formula**, which summarizes all of the experimental data:

$$\frac{1}{\lambda} = R_{\text{H}}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right),\ \text{where}\ n_i = n_f + 1,\ n_f + 2,\ n_f + 3,\ \ldots \tag{6.32}$$

When $n_f = 1,$ the series of spectral lines is called the **Lyman series**. When $n_f = 2,$ the series is called the Balmer series, and in this case, the Rydberg formula coincides with the Balmer formula. When $n_f = 3,$ the series is called the **Paschen series**. When $n_f = 4,$ the series is called the **Brackett series**. When $n_f = 5,$ the series is called the **Pfund series**. When $n_f = 6,$ we have the **Humphreys series**. As you may guess, there are infinitely many such spectral bands in the spectrum of hydrogen because n_f can be any positive integer number.

The Rydberg formula for hydrogen gives the exact positions of the spectral lines as they are observed in a laboratory; however, at the beginning of the twentieth century, nobody could explain why it worked so well. The Rydberg formula remained unexplained until the first successful model of the hydrogen atom was proposed in 1913.

Example 6.9

Limits of the Balmer Series

Calculate the longest and the shortest wavelengths in the Balmer series.

Strategy

We can use either the Balmer formula or the Rydberg formula. The longest wavelength is obtained when $1/n_i$ is largest, which is when $n_i = n_f + 1 = 3,$ because $n_f = 2$ for the Balmer series. The smallest wavelength is obtained when $1/n_i$ is smallest, which is $1/n_i \to 0$ when $n_i \to \infty$.

Solution

The long-wave limit:

$$\frac{1}{\lambda} = R_{\mathrm{H}}\left(\frac{1}{2^2} - \frac{1}{3^2}\right) = (1.09737 \times 10^7)\frac{1}{\mathrm{m}}\left(\frac{1}{4} - \frac{1}{9}\right) \Rightarrow \lambda = 656.3\ \mathrm{nm}$$

The short-wave limit:

$$\frac{1}{\lambda} = R_{\mathrm{H}}\left(\frac{1}{2^2} - 0\right) = (1.09737 \times 10^7)\frac{1}{\mathrm{m}}\left(\frac{1}{4}\right) \Rightarrow \lambda = 364.6\ \mathrm{nm}$$

Significance

Note that there are infinitely many spectral lines lying between these two limits.

6.9 Check Your Understanding What are the limits of the Lyman series? Can you see these spectral lines?

The key to unlocking the mystery of atomic spectra is in understanding atomic structure. Scientists have long known that matter is made of atoms. According to nineteenth-century science, atoms are the smallest indivisible quantities of matter. This scientific belief was shattered by a series of groundbreaking experiments that proved the existence of subatomic particles, such as electrons, protons, and neutrons.

The electron was discovered and identified as the smallest quantity of electric charge by J.J. Thomson in 1897 in his cathode ray experiments, also known as β-ray experiments: A ***β*-ray** is a beam of electrons. In 1904, Thomson proposed the first model of atomic structure, known as the “plum pudding” model, in which an atom consisted of an unknown positively charged matter with negative electrons embedded in it like plums in a pudding. Around 1900, E. Rutherford, and independently, Paul Ulrich Villard, classified all radiation known at that time as **α-rays**, β-rays, and **γ-rays** (a γ-ray is a beam of highly energetic photons). In 1907, Rutherford and Thomas Royds used spectroscopy methods to show that positively charged particles of α-radiation (called **α-particles**) are in fact doubly ionized atoms of helium. In 1909, Rutherford, Ernest Marsden, and Hans Geiger used α-particles in their famous scattering experiment that disproved Thomson’s model (see **Linear Momentum and Collisions (http://cnx.org/content/m58317/latest/)**).

In the **Rutherford gold foil experiment** (also known as the Geiger–Marsden experiment), α-particles were incident on a thin gold foil and were scattered by gold atoms inside the foil (see **m58321 (http://cnx.org/content/m58321/latest/#CNX_UPhysics_09_04_TvsR)**). The outgoing particles were detected by a $360°$ scintillation screen surrounding the gold target (for a detailed description of the experimental setup, see **Linear Momentum and Collisions (http://cnx.org/content/m58317/latest/)**). When a scattered particle struck the screen, a tiny flash of light (scintillation) was observed at that location. By counting the scintillations seen at various angles with respect to the direction of the incident beam, the scientists could determine what fraction of the incident particles were scattered and what fraction were not deflected at all. If the plum pudding model were correct, there would be no back-scattered α-particles. However, the results of the Rutherford experiment showed that, although a sizable fraction of α-particles emerged from the foil not scattered at all as though the foil were not in their way, a significant fraction of α-particles were back-scattered toward the source. This kind of result was possible only when most of the mass and the entire positive charge of the gold atom were concentrated in a tiny space inside the atom.

In 1911, Rutherford proposed a **nuclear model of the atom**. In Rutherford's model, an atom contained a positively charged nucleus of negligible size, almost like a point, but included almost the entire mass of the atom. The atom also contained negative electrons that were located within the atom but relatively far away from the nucleus. Ten years later, Rutherford coined the name *proton* for the nucleus of hydrogen and the name *neutron* for a hypothetical electrically neutral particle that would mediate the binding of positive protons in the nucleus (the neutron was discovered in 1932 by James Chadwick). Rutherford is credited with the discovery of the atomic nucleus; however, the Rutherford model of atomic structure does not explain the Rydberg formula for the hydrogen emission lines.

Bohr's model of the hydrogen atom, proposed by Niels Bohr in 1913, was the first quantum model that correctly explained the hydrogen emission spectrum. Bohr's model combines the classical mechanics of planetary motion with the quantum concept of photons. Once Rutherford had established the existence of the atomic nucleus, Bohr's intuition that the negative electron in the hydrogen atom must revolve around the positive nucleus became a logical consequence of the inverse-square-distance law of electrostatic attraction. Recall that Coulomb's law describing the attraction between two opposite charges has a similar form to Newton's universal law of gravitation in the sense that the gravitational force and the electrostatic force are both decreasing as $1/r^2$, where r is the separation distance between the bodies. In the same way as Earth revolves around the sun, the negative electron in the hydrogen atom can revolve around the positive nucleus. However, an accelerating charge radiates its energy. Classically, if the electron moved around the nucleus in a planetary fashion, it would be undergoing centripetal acceleration, and thus would be radiating energy that would cause it to spiral down into the nucleus. Such a planetary hydrogen atom would not be stable, which is contrary to what we know about ordinary hydrogen atoms that do not disintegrate. Moreover, the classical motion of the electron is not able to explain the discrete emission spectrum of hydrogen.

To circumvent these two difficulties, Bohr proposed the following three **postulates of Bohr's model**:

1. The negative electron moves around the positive nucleus (proton) in a circular orbit. All electron orbits are centered at the nucleus. Not all classically possible orbits are available to an electron bound to the nucleus.
2. The allowed electron orbits satisfy the *first quantization condition*: In the *n*th orbit, the angular momentum L_n of the electron can take only discrete values:

$$L_n = n\hbar, \text{ where } n = 1, 2, 3, \ldots \tag{6.33}$$

   This postulate says that the electron's angular momentum is quantized. Denoted by r_n and v_n, respectively, the radius of the *n*th orbit and the electron's speed in it, the first quantization condition can be expressed explicitly as

$$m_e v_n r_n = n\hbar. \tag{6.34}$$

3. An electron is allowed to make transitions from one orbit where its energy is E_n to another orbit where its energy is E_m. When an atom absorbs a photon, the electron makes a transition to a higher-energy orbit. When an atom emits a photon, the electron transits to a lower-energy orbit. Electron transitions with the simultaneous photon absorption or photon emission take place *instantaneously*. The allowed electron transitions satisfy the *second quantization condition*:

$$hf = |E_n - E_m| \tag{6.35}$$

   where hf is the energy of either an emitted or an absorbed photon with frequency *f*. The second quantization condition states that an electron's change in energy in the hydrogen atom is quantized.

These three postulates of the early quantum theory of the hydrogen atom allow us to derive not only the Rydberg formula, but also the value of the Rydberg constant and other important properties of the hydrogen atom such as its energy levels, its ionization energy, and the sizes of electron orbits. Note that in Bohr's model, along with two nonclassical quantization postulates, we also have the classical description of the electron as a particle that is subjected to the Coulomb force, and its motion must obey Newton's laws of motion. The hydrogen atom, as an isolated system, must obey the laws of conservation of energy and momentum in the way we know from classical physics. Having this theoretical framework in mind, we are ready to proceed with our analysis.

Electron Orbits

To obtain the size r_n of the electron's *n*th orbit and the electron's speed v_n in it, we turn to Newtonian mechanics. As a charged particle, the electron experiences an electrostatic pull toward the positively charged nucleus in the center of its circular orbit. This electrostatic pull is the centripetal force that causes the electron to move in a circle around the nucleus. Therefore, the magnitude of centripetal force is identified with the magnitude of the electrostatic force:

$$\frac{m_e v_n^2}{r_n} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_n^2}. \quad (6.36)$$

Here, e denotes the value of the elementary charge. The negative electron and positive proton have the same value of charge, $|q| = e$. When **Equation 6.36** is combined with the first quantization condition given by **Equation 6.34**, we can solve for the speed, v_n, and for the radius, r_n:

$$v_n = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar}\frac{1}{n} \quad (6.37)$$

$$r_n = 4\pi\varepsilon_0 \frac{\hbar^2}{m_e e^2} n^2. \quad (6.38)$$

Note that these results tell us that the electron's speed as well as the radius of its orbit depend only on the index *n* that enumerates the orbit because all other quantities in the preceding equations are fundamental constants. We see from **Equation 6.38** that the size of the orbit grows as the square of *n*. This means that the second orbit is four times as large as the first orbit, and the third orbit is nine times as large as the first orbit, and so on. We also see from **Equation 6.37** that the electron's speed in the orbit decreases as the orbit size increases. The electron's speed is largest in the first Bohr orbit, for $n = 1$, which is the orbit closest to the nucleus. The radius of the first Bohr orbit is called the **Bohr radius of hydrogen**, denoted as a_0. Its value is obtained by setting $n = 1$ in **Equation 6.38**:

$$a_0 = 4\pi\varepsilon_0 \frac{\hbar^2}{m_e e^2} = 5.29 \times 10^{-11}\,\text{m} = 0.529\,\text{Å}. \quad (6.39)$$

We can substitute a_0 in **Equation 6.38** to express the radius of the *n*th orbit in terms of a_0:

$$r_n = a_0 n^2. \quad (6.40)$$

This result means that the electron orbits in hydrogen atom are *quantized* because the orbital radius takes on only specific values of $a_0, 4a_0, 9a_0, 16a_0, \ldots$ given by **Equation 6.40**, and no other values are allowed.

Electron Energies

The total energy E_n of an electron in the *n*th orbit is the sum of its kinetic energy K_n and its electrostatic potential energy U_n. Utilizing **Equation 6.37**, we find that

$$K_n = \frac{1}{2} m_e v_n^2 = \frac{1}{32\pi^2\varepsilon_0^2}\frac{m_e e^4}{\hbar^2}\frac{1}{n^2}. \quad (6.41)$$

Recall that the electrostatic potential energy of interaction between two charges q_1 and q_2 that are separated by a distance r_{12} is $(1/4\pi\varepsilon_0)q_1 q_2/r_{12}$. Here, $q_1 = +e$ is the charge of the nucleus in the hydrogen atom (the charge of the proton),

$q_2 = -e$ is the charge of the electron and $r_{12} = r_n$ is the radius of the *n*th orbit. Now we use **Equation 6.38** to find the potential energy of the electron:

$$U_n = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_n} = -\frac{1}{16\pi^2\varepsilon_0^2}\frac{m_e e^4}{\hbar^2}\frac{1}{n^2}. \tag{6.42}$$

The total energy of the electron is the sum of **Equation 6.41** and **Equation 6.42**:

$$E_n = K_n + U_n = -\frac{1}{32\pi^2\varepsilon_0^2}\frac{m_e e^4}{\hbar^2}\frac{1}{n^2}. \tag{6.43}$$

Note that the energy depends only on the index *n* because the remaining symbols in **Equation 6.43** are physical constants. The value of the constant factor in **Equation 6.43** is

$$E_0 = \frac{1}{32\pi^2\varepsilon_0^2}\frac{m_e e^4}{\hbar^2} = \frac{1}{8\varepsilon_0^2}\frac{m_e e^4}{h^2} = 2.17 \times 10^{-18}\,\text{J} = 13.6\,\text{eV}. \tag{6.44}$$

It is convenient to express the electron's energy in the *n*th orbit in terms of this energy, as

$$E_n = -E_0\frac{1}{n^2}. \tag{6.45}$$

Now we can see that the electron energies in the hydrogen atom are *quantized* because they can have only discrete values of $-E_0,\ -E_0/4,\ -E_0/9,\ -E_0/16,\ ...$ given by **Equation 6.45**, and no other energy values are allowed. This set of allowed electron energies is called the **energy spectrum of hydrogen** (**Figure 6.17**). The index *n* that enumerates energy levels in Bohr's model is called the energy **quantum number**. We identify the energy of the electron inside the hydrogen atom with the energy of the hydrogen atom. Note that the smallest value of energy is obtained for $n = 1,$ so the hydrogen atom cannot have energy smaller than that. This smallest value of the electron energy in the hydrogen atom is called the **ground state energy of the hydrogen atom** and its value is

$$E_1 = -E_0 = -13.6\,\text{eV}. \tag{6.46}$$

The hydrogen atom may have other energies that are higher than the ground state. These higher energy states are known as **excited energy states of a hydrogen atom**.

There is only one ground state, but there are infinitely many excited states because there are infinitely many values of *n* in **Equation 6.45**. We say that the electron is in the "first exited state" when its energy is E_2 (when $n = 2$), the second excited state when its energy is E_3 (when $n = 3$) and, in general, in the *n*th exited state when its energy is E_{n+1}. There is no highest-of-all excited state; however, there is a limit to the sequence of excited states. If we keep increasing *n* in **Equation 6.45**, we find that the limit is $-\lim_{n\to\infty} E_0/n^2 = 0.$ In this limit, the electron is no longer bound to the nucleus but becomes a free electron. An electron remains bound in the hydrogen atom as long as its energy is negative. An electron that orbits the nucleus in the first Bohr orbit, closest to the nucleus, is in the ground state, where its energy has the smallest value. In the ground state, the electron is most strongly bound to the nucleus and its energy is given by **Equation 6.46**. If we want to remove this electron from the atom, we must supply it with enough energy, $E_\infty,$ to at least balance out its ground state energy E_1:

$$E_\infty + E_1 = 0 \Rightarrow E_\infty = -E_1 = -(-E_0) = E_0 = 13.6\,\text{eV}. \tag{6.47}$$

The energy that is needed to remove the electron from the atom is called the **ionization energy**. The ionization energy E_∞ that is needed to remove the electron from the first Bohr orbit is called the **ionization limit of the hydrogen atom**. The ionization limit in **Equation 6.47** that we obtain in Bohr's model agrees with experimental value.

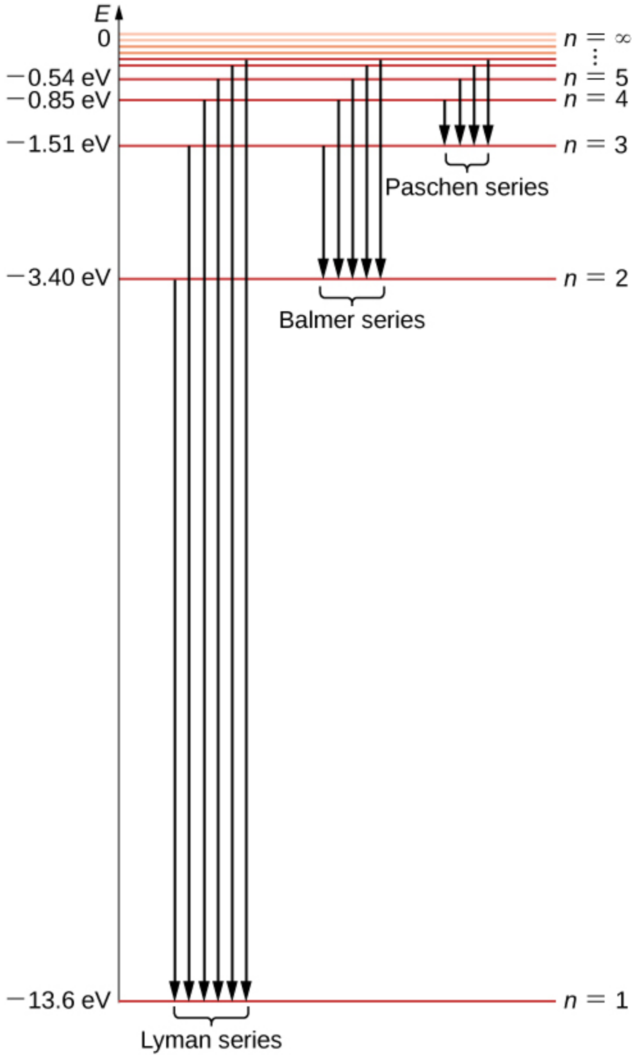


Figure 6.17 The energy spectrum of the hydrogen atom. Energy levels (horizontal lines) represent the bound states of an electron in the atom. There is only one ground state, $n = 1$, and infinite quantized excited states. The states are enumerated by the quantum number $n = 1, 2, 3, 4, ...$ Vertical lines illustrate the allowed electron transitions between the states. Downward arrows illustrate transitions with an emission of a photon with a wavelength in the indicated spectral band.

Spectral Emission Lines of Hydrogen

To obtain the wavelengths of the emitted radiation when an electron makes a transition from the *n*th orbit to the *m*th orbit, we use the second of Bohr's quantization conditions and **Equation 6.45** for energies. The emission of energy from the atom can occur only when an electron makes a transition from an excited state to a lower-energy state. In the course of such a transition, the emitted photon carries away the difference of energies between the states involved in the transition. The transition cannot go in the other direction because the energy of a photon cannot be negative, which means that for emission we must have $E_n > E_m$ and $n > m$. Therefore, the third of Bohr's postulates gives

$$hf = \left|E_n - E_m\right| = E_n - E_m = -E_0\frac{1}{n^2} + E_0\frac{1}{m^2} = E_0\left(\frac{1}{m^2} - \frac{1}{n^2}\right). \quad (6.48)$$

Now we express the photon's energy in terms of its wavelength, $hf = hc/\lambda$, and divide both sides of **Equation 6.48** by hc. The result is

$$\frac{1}{\lambda} = \frac{E_0}{hc}\left(\frac{1}{m^2} - \frac{1}{n^2}\right). \tag{6.49}$$

The value of the constant in this equation is

$$\frac{E_0}{hc} = \frac{13.6\,\text{eV}}{(4.136 \times 10^{-15}\,\text{eV}\cdot\text{s})(2.997 \times 10^8\,\text{m/s})} = 1.097 \times 10^7\,\frac{1}{\text{m}}. \tag{6.50}$$

This value is exactly the Rydberg constant R_{H} in the Rydberg heuristic formula **Equation 6.32**. In fact, **Equation 6.49** is identical to the Rydberg formula, because for a given *m*, we have $n = m+1,\ m+2,\ \ldots.$ In this way, the Bohr quantum model of the hydrogen atom allows us to derive the experimental Rydberg constant from first principles and to express it in terms of fundamental constants. Transitions between the allowed electron orbits are illustrated in **Figure 6.17**.

We can repeat the same steps that led to **Equation 6.49** to obtain the wavelength of the absorbed radiation; this again gives **Equation 6.49** but this time for the positions of absorption lines in the absorption spectrum of hydrogen. The only difference is that for absorption, the quantum number *m* is the index of the orbit occupied by the electron before the transition (lower-energy orbit) and the quantum number *n* is the index of the orbit to which the electron makes the transition (higher-energy orbit). The difference between the electron energies in these two orbits is the energy of the absorbed photon.

Example 6.10

Size and Ionization Energy of the Hydrogen Atom in an Excited State

If a hydrogen atom in the ground state absorbs a 93.7-nm photon, corresponding to a transition line in the Lyman series, how does this affect the atom's energy and size? How much energy is needed to ionize the atom when it is in this excited state? Give your answers in absolute units, and relative to the ground state.

Strategy

Before the absorption, the atom is in its ground state. This means that the electron transition takes place from the orbit $m = 1$ to some higher *n*th orbit. First, we must determine n for the absorbed wavelength $\lambda = 93.7\,\text{nm}$. Then, we can use **Equation 6.45** to find the energy E_n of the excited state and its ionization energy $E_{\infty,\,n}$, and use **Equation 6.40** to find the radius r_n of the atom in the excited state. To estimate *n*, we use **Equation 6.49**.

Solution

Substitute $m = 1$ and $\lambda = 93.7\,\text{nm}$ in **Equation 6.49** and solve for n. You should not expect to obtain a perfect integer answer because of rounding errors, but your answer will be close to an integer, and you can estimate *n* by taking the integral part of your answer:

$$\frac{1}{\lambda} = R_{\text{H}}\left(\frac{1}{1^2} - \frac{1}{n^2}\right) \Rightarrow n = \frac{1}{\sqrt{1 - \frac{1}{\lambda R_{\text{H}}}}} = \frac{1}{\sqrt{1 - \frac{1}{(93.7 \times 10^{-9}\,\text{m})(1.097 \times 10^7\,\text{m}^{-1})}}} = 6.07 \Rightarrow n = 6.$$

The radius of the $n = 6$ orbit is

$$r_n = a_0 n^2 = a_0 6^2 = 36a_0 = 36(0.529 \times 10^{-10}\,\text{m}) = 19.04 \times 10^{-10}\,\text{m} \cong 19.0\,\text{Å}.$$

Thus, after absorbing the 93.7-nm photon, the size of the hydrogen atom in the excited $n = 6$ state is 36 times larger than before the absorption, when the atom was in the ground state. The energy of the fifth excited state ($n = 6$) is:

$$E_n = -\frac{E_0}{n^2} = -\frac{E_0}{6^2} = -\frac{E_0}{36} = -\frac{13.6\,\text{eV}}{36} \cong -0.378\,\text{eV}.$$

After absorbing the 93.7-nm photon, the energy of the hydrogen atom is larger than it was before the absorption. Ionization of the atom when it is in the fifth excited state ($n = 6$) requites 36 times less energy than is needed when the atom is in the ground state:

$$E_{\infty,\,6} = -E_6 = -(-0.378\text{ eV}) = 0.378\text{ eV}.$$

Significance

We can analyze any spectral line in the spectrum of hydrogen in the same way. Thus, the experimental measurements of spectral lines provide us with information about the atomic structure of the hydrogen atom.

6.10 Check Your Understanding When an electron in a hydrogen atom is in the first excited state, what prediction does the Bohr model give about its orbital speed and kinetic energy? What is the magnitude of its orbital angular momentum?

Bohr's model of the hydrogen atom also correctly predicts the spectra of some hydrogen-like ions. **Hydrogen-like ions** are atoms of elements with an atomic number Z larger than one ($Z = 1$ for hydrogen) but with all electrons removed except one. For example, an electrically neutral helium atom has an atomic number $Z = 2$. This means it has two electrons orbiting the nucleus with a charge of $q = +Ze$. When one of the orbiting electrons is removed from the helium atom (we say, when the helium atom is singly ionized), what remains is a hydrogen-like atomic structure where the remaining electron orbits the nucleus with a charge of $q = +Ze$. This type of situation is described by the Bohr model. Assuming that the charge of the nucleus is not $+e$ but $+Ze$, we can repeat all steps, beginning with **Equation 6.36**, to obtain the results for a hydrogen-like ion:

$$r_n = \frac{a_0}{Z} n^2 \qquad (6.51)$$

where a_0 is the Bohr orbit of hydrogen, and

$$E_n = -Z^2 E_0 \frac{1}{n^2} \qquad (6.52)$$

where E_0 is the ionization limit of a hydrogen atom. These equations are good approximations as long as the atomic number Z is not too large.

The Bohr model is important because it was the first model to postulate the quantization of electron orbits in atoms. Thus, it represents an early quantum theory that gave a start to developing modern quantum theory. It introduced the concept of a quantum number to describe atomic states. The limitation of the early quantum theory is that it cannot describe atoms in which the number of electrons orbiting the nucleus is larger than one. The Bohr model of hydrogen is a semi-classical model because it combines the classical concept of electron orbits with the new concept of quantization. The remarkable success of this model prompted many physicists to seek an explanation for why such a model should work at all, and to seek an understanding of the physics behind the postulates of early quantum theory. This search brought about the onset of an entirely new concept of "matter waves."

6.5 | De Broglie's Matter Waves

Learning Objectives

By the end of this section, you will be able to:

- Describe de Broglie's hypothesis of matter waves
- Explain how the de Broglie's hypothesis gives the rationale for the quantization of angular momentum in Bohr's quantum theory of the hydrogen atom
- Describe the Davisson–Germer experiment
- Interpret de Broglie's idea of matter waves and how they account for electron diffraction phenomena

Compton's formula established that an electromagnetic wave can behave like a particle of light when interacting with matter. In 1924, Louis de Broglie proposed a new speculative hypothesis that electrons and other particles of matter can behave like waves. Today, this idea is known as **de Broglie's hypothesis of matter waves**. In 1926, De Broglie's hypothesis, together with Bohr's early quantum theory, led to the development of a new theory of **wave quantum mechanics** to describe the physics of atoms and subatomic particles. Quantum mechanics has paved the way for new engineering inventions and technologies, such as the laser and magnetic resonance imaging (MRI). These new technologies drive discoveries in other sciences such as biology and chemistry.

According to de Broglie's hypothesis, massless photons as well as massive particles must satisfy one common set of relations that connect the energy E with the frequency f, and the linear momentum p with the wavelength λ. We have discussed these relations for photons in the context of Compton's effect. We are recalling them now in a more general context. Any particle that has energy and momentum is a **de Broglie wave** of frequency f and wavelength λ :

$$E = hf \tag{6.53}$$

$$\lambda = \frac{h}{p}. \tag{6.54}$$

Here, E and p are, respectively, the relativistic energy and the momentum of a particle. De Broglie's relations are usually expressed in terms of the wave vector $\vec{\mathbf{k}}$, $k = 2\pi/\lambda$, and the wave frequency $\omega = 2\pi f$, as we usually do for waves:

$$E = \hbar\omega \tag{6.55}$$

$$\vec{\mathbf{p}} = \hbar\vec{\mathbf{k}}. \tag{6.56}$$

Wave theory tells us that a wave carries its energy with the **group velocity**. For matter waves, this group velocity is the velocity u of the particle. Identifying the energy E and momentum p of a particle with its relativistic energy mc^2 and its relativistic momentum mu, respectively, it follows from de Broglie relations that matter waves satisfy the following relation:

$$\lambda f = \frac{\omega}{k} = \frac{E/\hbar}{p/\hbar} = \frac{E}{p} = \frac{mc^2}{mu} = \frac{c^2}{u} = \frac{c}{\beta} \tag{6.57}$$

where $\beta = u/c$. When a particle is massless we have $u = c$ and Equation 6.57 becomes $\lambda f = c$.

Example 6.11

How Long Are de Broglie Matter Waves?

Calculate the de Broglie wavelength of: (a) a 0.65-kg basketball thrown at a speed of 10 m/s, (b) a nonrelativistic electron with a kinetic energy of 1.0 eV, and (c) a relativistic electron with a kinetic energy of $108\,\text{keV}$.

Strategy

We use **Equation 6.57** to find the de Broglie wavelength. When the problem involves a nonrelativistic object moving with a nonrelativistic speed *u*, such as in (a) when $\beta = u/c \ll 1,$ we use nonrelativistic momentum *p*. When the nonrelativistic approximation cannot be used, such as in (c), we must use the relativistic momentum $p = mu = m_0 \gamma u = E_0 \gamma \beta,$ where the rest mass energy of a particle is $E_0 = m_0 c^2$ and γ is the Lorentz factor $\gamma = 1/\sqrt{1-\beta^2}$. The total energy *E* of a particle is given by **Equation 6.53** and the kinetic energy is $K = E - E_0 = (\gamma - 1)E_0$. When the kinetic energy is known, we can invert **Equation 6.18** to find the momentum $p = \sqrt{(E^2 - E_0^2)/c^2} = \sqrt{K(K+2E_0)}/c$ and substitute in **Equation 6.57** to obtain

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{K(K+2E_0)}}. \qquad (6.58)$$

Depending on the problem at hand, in this equation we can use the following values for *hc*: $hc = (6.626 \times 10^{-34}\,\text{J}\cdot\text{s})(2.998 \times 10^8\,\text{m/s}) = 1.986 \times 10^{-25}\,\text{J}\cdot\text{m} = 1.241\,\text{eV}\cdot\mu\text{m}$

Solution

a. For the basketball, the kinetic energy is

$$K = m_0 u^2/2 = (0.65\text{kg})(10\text{m/s})^2/2 = 32.5\text{J}$$

and the rest mass energy is

$$E_0 = m_0 c^2 = (0.65\text{kg})(2.998 \times 10^8\,\text{m/s})^2 = 5.84 \times 10^{16}\,\text{J}.$$

We see that $K/(K+E_0) \ll 1$ and use $p = m_0 u = (0.65\text{kg})(10\text{m/s}) = 6.5\,\text{J}\cdot\text{s/m}$:

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34}\,\text{J}\cdot\text{s}}{6.5\text{J}\cdot\text{s/m}} = 1.02 \times 10^{-34}\,\text{m}.$$

b. For the nonrelativistic electron,

$$E_0 = m_0 c^2 = (9.109 \times 10^{-31}\,\text{kg})(2.998 \times 10^8\,\text{m/s})^2 = 511\,\text{keV}$$

and when $K = 1.0\,\text{eV},$ we have $K/(K+E_0) = (1/512) \times 10^{-3} \ll 1,$ so we can use the nonrelativistic formula. However, it is simpler here to use **Equation 6.58**:

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{K(K+2E_0)}} = \frac{1.241\,\text{eV}\cdot\mu\text{m}}{\sqrt{(1.0\,\text{eV})[1.0\,\text{eV}+2(511\,\text{keV})]}} = 1.23\,\text{nm}.$$

If we use nonrelativistic momentum, we obtain the same result because 1 eV is much smaller than the rest mass of the electron.

c. For a fast electron with $K = 108\,\text{keV},$ relativistic effects cannot be neglected because its total energy is $E = K + E_0 = 108\,\text{keV} + 511\,\text{keV} = 619\,\text{keV}$ and $K/E = 108/619$ is not negligible:

$$\lambda = \frac{h}{p} = \frac{hc}{\sqrt{K(K+2E_0)}} = \frac{1.241\,\text{eV}\cdot\mu\text{m}}{\sqrt{108\,\text{keV}[108\,\text{keV} + 2(511\,\text{keV})]}} = 3.55\,\text{pm}.$$

Significance

We see from these estimates that De Broglie's wavelengths of macroscopic objects such as a ball are immeasurably small. Therefore, even if they exist, they are not detectable and do not affect the motion of macroscopic objects.

6.11 Check Your Understanding What is de Broglie's wavelength of a nonrelativistic proton with a kinetic energy of 1.0 eV?

Using the concept of the electron matter wave, de Broglie provided a rationale for the quantization of the electron's angular momentum in the hydrogen atom, which was postulated in Bohr's quantum theory. The physical explanation for the first Bohr quantization condition comes naturally when we assume that an electron in a hydrogen atom behaves not like a particle but like a wave. To see it clearly, imagine a stretched guitar string that is clamped at both ends and vibrates in one of its normal modes. If the length of the string is l (**Figure 6.18**), the wavelengths of these vibrations cannot be arbitrary but must be such that an integer k number of half-wavelengths $\lambda/2$ fit exactly on the distance l between the ends. This is the condition $l = k\lambda/2$ for a standing wave on a string. Now suppose that instead of having the string clamped at the walls, we bend its length into a circle and fasten its ends to each other. This produces a circular string that vibrates in normal modes, satisfying the same standing-wave condition, but the number of half-wavelengths must now be an even number k, $k = 2n$, and the length l is now connected to the radius r_n of the circle. This means that the radii are not arbitrary but must satisfy the following standing-wave condition:

$$2\pi r_n = 2n\frac{\lambda}{2}. \tag{6.59}$$

If an electron in the nth Bohr orbit moves as a wave, by **Equation 6.59** its wavelength must be equal to $\lambda = 2\pi r_n/n$. Assuming that **Equation 6.58** is valid, the electron wave of this wavelength corresponds to the electron's linear momentum, $p = h/\lambda = nh/(2\pi r_n) = n\hbar/r_n$. In a circular orbit, therefore, the electron's angular momentum must be

$$L_n = r_n p = r_n \frac{n\hbar}{r_n} = n\hbar. \tag{6.60}$$

This equation is the first of Bohr's quantization conditions, given by **Equation 6.36**. Providing a physical explanation for Bohr's quantization condition is a convincing theoretical argument for the existence of matter waves.

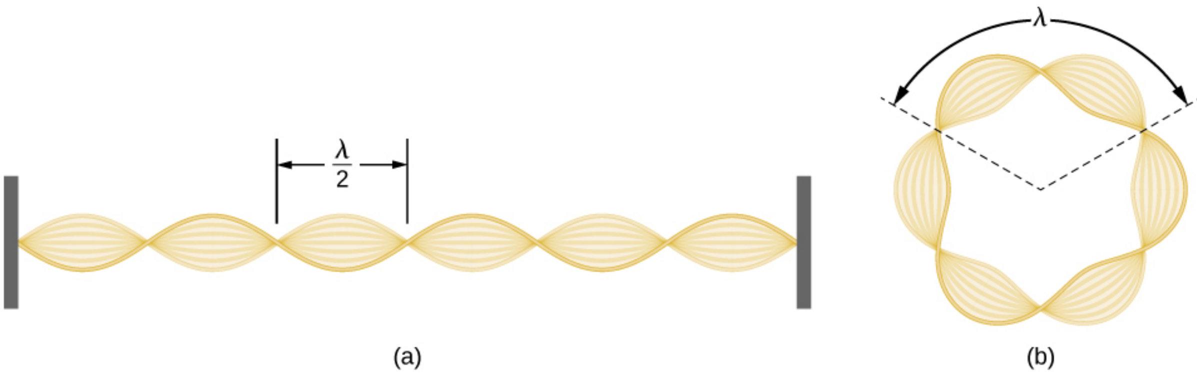


Figure 6.18 Standing-wave pattern: (a) a stretched string clamped at the walls; (b) an electron wave trapped in the third Bohr orbit in the hydrogen atom.

Example 6.12

The Electron Wave in the Ground State of Hydrogen

Find the de Broglie wavelength of an electron in the ground state of hydrogen.

Strategy

We combine the first quantization condition in **Equation 6.60** with **Equation 6.36** and use **Equation 6.38** for the first Bohr radius with $n = 1$.

Solution

When $n = 1$ and $r_n = a_0 = 0.529$ Å, the Bohr quantization condition gives $a_0 p = 1 \cdot \hbar \Rightarrow p = \hbar / a_0$. The electron wavelength is:

$$\lambda = h/p = h/\hbar/a_0 = 2\pi a_0 = 2\pi(0.529 \text{ Å}) = 3.324 \text{ Å}.$$

Significance

We obtain the same result when we use **Equation 6.58** directly.

6.12 Check Your Understanding Find the de Broglie wavelength of an electron in the third excited state of hydrogen.

Experimental confirmation of matter waves came in 1927 when C. Davisson and L. Germer performed a series of electron-scattering experiments that clearly showed that electrons do behave like waves. Davisson and Germer did not set up their experiment to confirm de Broglie's hypothesis: The confirmation came as a byproduct of their routine experimental studies of metal surfaces under electron bombardment.

In the particular experiment that provided the very first evidence of electron waves (known today as the **Davisson–Germer experiment**), they studied a surface of nickel. Their nickel sample was specially prepared in a high-temperature oven to change its usual polycrystalline structure to a form in which large single-crystal domains occupy the volume. **Figure 6.19** shows the experimental setup. Thermal electrons are released from a heated element (usually made of tungsten) in the electron gun and accelerated through a potential difference ΔV, becoming a well-collimated beam of electrons produced by an electron gun. The kinetic energy *K* of the electrons is adjusted by selecting a value of the potential difference in the electron gun. This produces a beam of electrons with a set value of linear momentum, in accordance with the conservation of energy:

$$e\Delta V = K = \frac{p^2}{2m} \Rightarrow p = \sqrt{2me\Delta V}. \quad (6.61)$$

The electron beam is incident on the nickel sample in the direction normal to its surface. At the surface, it scatters in various directions. The intensity of the beam scattered in a selected direction φ is measured by a highly sensitive detector. The detector's angular position with respect to the direction of the incident beam can be varied from $\varphi = 0°$ to $\varphi = 90°$. The entire setup is enclosed in a vacuum chamber to prevent electron collisions with air molecules, as such thermal collisions would change the electrons' kinetic energy and are not desirable.

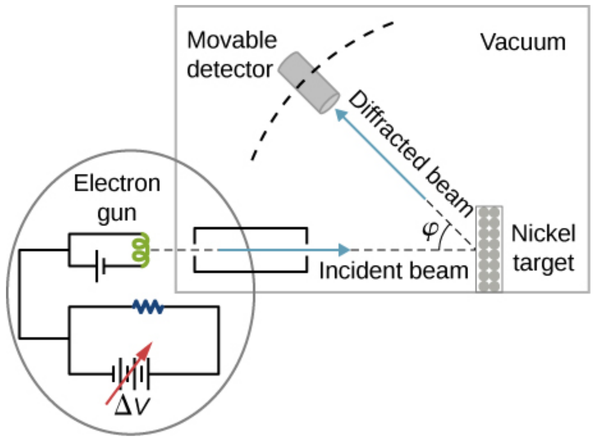


Figure 6.19 Schematics of the experimental setup of the Davisson–Germer diffraction experiment. A well-collimated beam of electrons is scattered off the nickel target. The kinetic energy of electrons in the incident beam is selected by adjusting a variable potential, ΔV, in the electron gun. Intensity of the scattered electron beam is measured for a range of scattering angles φ, whereas the distance between the detector and the target does not change.

When the nickel target has a polycrystalline form with many randomly oriented microscopic crystals, the incident electrons scatter off its surface in various random directions. As a result, the intensity of the scattered electron beam is much the same in any direction, resembling a diffuse reflection of light from a porous surface. However, when the nickel target has a regular crystalline structure, the intensity of the scattered electron beam shows a clear maximum at a specific angle and the results show a clear diffraction pattern (see **Figure 6.20**). Similar diffraction patterns formed by X-rays scattered by various crystalline solids were studied in 1912 by father-and-son physicists William H. Bragg and William L. Bragg. The Bragg law in X-ray crystallography provides a connection between the wavelength λ of the radiation incident on a crystalline lattice, the lattice spacing, and the position of the interference maximum in the diffracted radiation (see **Diffraction**).

The lattice spacing of the Davisson–Germer target, determined with X-ray crystallography, was measured to be $a = 2.15\,\text{Å}$. Unlike X-ray crystallography in which X-rays penetrate the sample, in the original Davisson–Germer experiment, only the surface atoms interact with the incident electron beam. For the surface diffraction, the maximum intensity of the reflected electron beam is observed for scattering angles that satisfy the condition $n\lambda = a\sin\varphi$ (see **Figure 6.21**). The first-order maximum (for $n = 1$) is measured at a scattering angle of $\varphi \approx 50°$ at $\Delta V \approx 54\text{V}$, which gives the wavelength of the incident radiation as $\lambda = (2.15\,\text{Å})\sin 50° = 1.64\,\text{Å}$. On the other hand, a 54-V potential accelerates the incident electrons to kinetic energies of $K = 54\,\text{eV}$. Their momentum, calculated from **Equation 6.61**, is $p = 2.478 \times 10^{-5}\,\text{eV} \cdot \text{s/m}$. When we substitute this result in **Equation 6.58**, the de Broglie wavelength is obtained as

$$\lambda = \frac{h}{p} = \frac{4.136 \times 10^{-15}\,\text{eV} \cdot \text{s}}{2.478 \times 10^{-5}\,\text{eV} \cdot \text{s/m}} = 1.67\,\text{Å}. \tag{6.62}$$

The same result is obtained when we use $K = 54\,\text{eV}$ in **Equation 6.61**. The proximity of this theoretical result to the Davisson–Germer experimental value of $\lambda = 1.64\,\text{Å}$ is a convincing argument for the existence of de Broglie matter waves.

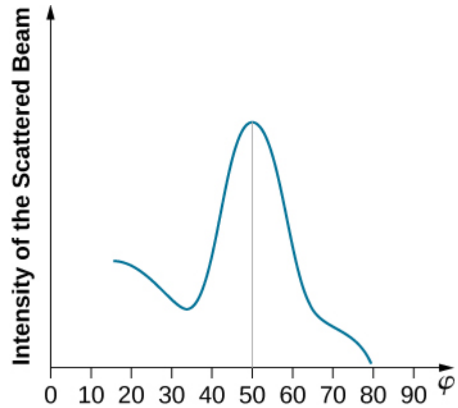
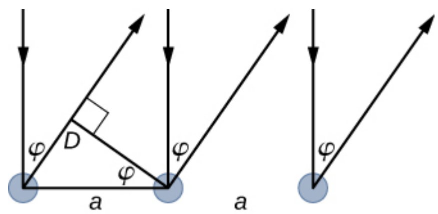


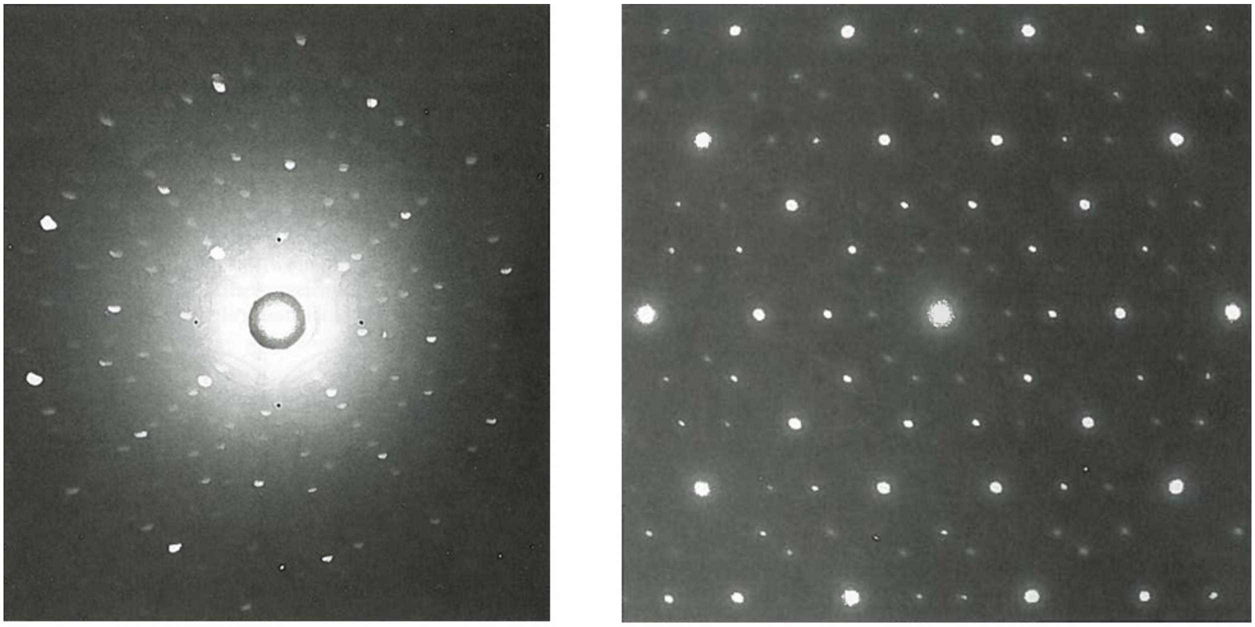
Figure 6.20 The experimental results of electron diffraction on a nickel target for the accelerating potential in the electron gun of about $\Delta V = 54\text{V}$: The intensity maximum is registered at the scattering angle of about $\varphi = 50°$.



$$D = a \sin \varphi$$
$$D = n\lambda \quad n = 1, 2, 3, \ldots$$
$$n\lambda = a \sin \varphi$$

Figure 6.21 In the surface diffraction of a monochromatic electromagnetic wave on a crystalline lattice structure, the in-phase incident beams are reflected from atoms on the surface. A ray reflected from the left atom travels an additional distance $D = a\sin\theta$ to the detector, where *a* is the lattice spacing. The reflected beams remain in-phase when *D* is an integer multiple of their wavelength λ. The intensity of the reflected waves has pronounced maxima for angles φ satisfying $n\lambda = a\sin\varphi$.

Diffraction lines measured with low-energy electrons, such as those used in the Davisson–Germer experiment, are quite broad (see **Figure 6.20**) because the incident electrons are scattered only from the surface. The resolution of diffraction images greatly improves when a higher-energy electron beam passes through a thin metal foil. This occurs because the diffraction image is created by scattering off many crystalline planes inside the volume, and the maxima produced in scattering at Bragg angles are sharp (see **Figure 6.22**).



(a) (b)

Figure 6.22 Diffraction patterns obtained in scattering on a crystalline solid: (a) with X-rays, and (b) with electrons. The observed pattern reflects the symmetry of the crystalline structure of the sample.

Since the work of Davisson and Germer, de Broglie's hypothesis has been extensively tested with various experimental techniques, and the existence of de Broglie waves has been confirmed for numerous elementary particles. Neutrons have been used in scattering experiments to determine crystalline structures of solids from interference patterns formed by neutron matter waves. The neutron has zero charge and its mass is comparable with the mass of a positively charged proton. Both neutrons and protons can be seen as matter waves. Therefore, the property of being a matter wave is not specific to electrically charged particles but is true of all particles in motion. Matter waves of molecules as large as carbon C_{60} have been measured. All physical objects, small or large, have an associated matter wave as long as they remain in motion. The universal character of de Broglie matter waves is firmly established.

Example 6.13

Neutron Scattering

Suppose that a neutron beam is used in a diffraction experiment on a typical crystalline solid. Estimate the kinetic energy of a neutron (in eV) in the neutron beam and compare it with kinetic energy of an ideal gas in equilibrium at room temperature.

Strategy

We assume that a typical crystal spacing a is of the order of 1.0 Å. To observe a diffraction pattern on such a lattice, the neutron wavelength λ must be on the same order of magnitude as the lattice spacing. We use **Equation 6.61** to find the momentum p and kinetic energy K. To compare this energy with the energy E_T of ideal gas in equilibrium at room temperature $T = 300\text{K}$, we use the relation $K = \frac{3}{2}k_B T$, where $k_B = 8.62 \times 10^{-5}\,\text{eV/K}$ is the Boltzmann constant.

Solution

We evaluate pc to compare it with the neutron's rest mass energy $E_0 = 940\,\text{MeV}$:

$$p = \frac{h}{\lambda} \Rightarrow pc = \frac{hc}{\lambda} = \frac{1.241 \times 10^{-6}\,\text{eV}\cdot\text{m}}{10^{-10}\,\text{m}} = 12.41\,\text{keV}.$$

We see that $p^2 c^2 \ll E_0^2$ so $K \ll E_0$ and we can use the nonrelativistic kinetic energy:

$$K = \frac{p^2}{2m_n} = \frac{h^2}{2\lambda^2 m_n} = \frac{(6.63 \times 10^{-34}\,\text{J}\cdot\text{s})^2}{(2 \times 10^{-20}\,\text{m}^2)(1.66 \times 10^{-27}\,\text{kg})} = 1.32 \times 10^{-20}\,\text{J} = 82.7\,\text{meV}.$$

Kinetic energy of ideal gas in equilibrium at 300 K is:

$$K_T = \frac{3}{2}k_B T = \frac{3}{2}(8.62 \times 10^{-5}\,\text{eV/K})(300\text{K}) = 38.8\,\text{MeV}.$$

We see that these energies are of the same order of magnitude.

Significance

Neutrons with energies in this range, which is typical for an ideal gas at room temperature, are called “thermal neutrons.”

Example 6.14

Wavelength of a Relativistic Proton

In a supercollider at CERN, protons can be accelerated to velocities of 0.75*c*. What are their de Broglie wavelengths at this speed? What are their kinetic energies?

Strategy

The rest mass energy of a proton is $E_0 = m_0 c^2 = (1.672 \times 10^{-27}\,\text{kg})(2.998 \times 10^8\,\text{m/s})^2 = 938\,\text{MeV}$. When the proton’s velocity is known, we have $\beta = 0.75$ and $\beta\gamma = 0.75/\sqrt{1 - 0.75^2} = 1.714$. We obtain the wavelength λ and kinetic energy K from relativistic relations.

Solution

$$\lambda = \frac{h}{p} = \frac{hc}{pc} = \frac{hc}{\beta\gamma E_0} = \frac{1.241\,\text{eV}\cdot\mu\text{m}}{1.714(938\,\text{MeV})} = 0.77\,\text{fm}$$

$$K = E_0(\gamma - 1) = 938\,\text{MeV}(1/\sqrt{1 - 0.75^2} - 1) = 480.1\,\text{MeV}$$

Significance

Notice that because a proton is 1835 times more massive than an electron, if this experiment were performed with electrons, a simple rescaling of these results would give us the electron’s wavelength of $(1835)0.77\text{fm} = 1.4\,\text{pm}$ and its kinetic energy of $480.1\,\text{MeV}/1835 = 261.6\,\text{keV}$.

6.13 Check Your Understanding Find the de Broglie wavelength and kinetic energy of a free electron that travels at a speed of 0.75*c*.

6.6 | Wave-Particle Duality

Learning Objectives

By the end of this section, you will be able to:

- Identify phenomena in which electromagnetic waves behave like a beam of photons and particles behave like waves
- Describe the physics principles behind electron microscopy
- Summarize the evolution of scientific thought that led to the development of quantum mechanics

The energy of radiation detected by a radio-signal receiving antenna comes as the energy of an electromagnetic wave. The same energy of radiation detected by a photocurrent in the photoelectric effect comes as the energy of individual photon particles. Therefore, the question arises about the nature of electromagnetic radiation: Is a photon a wave or is it a particle? Similar questions may be asked about other known forms of energy. For example, an electron that forms part of an electric current in a circuit behaves like a particle moving in unison with other electrons inside the conductor. The same electron behaves as a wave when it passes through a solid crystalline structure and forms a diffraction image. Is an electron a wave or is it a particle? The same question can be extended to all particles of matter—elementary particles, as well as compound molecules—asking about their true physical nature. At our present state of knowledge, such questions about the true nature of things do not have conclusive answers. All we can say is that **wave-particle duality** exists in nature: Under some experimental conditions, a particle appears to act as a particle, and under different experimental conditions, a particle appears to act a wave. Conversely, under some physical circumstances electromagnetic radiation acts as a wave, and under other physical circumstances, radiation acts as a beam of photons.

This dualistic interpretation is not a new physics concept brought about by specific discoveries in the twentieth century. It was already present in a debate between Isaac Newton and Christiaan Huygens about the nature of light, beginning in the year 1670. According to Newton, a beam of light is a collection of corpuscles of light. According to Huygens, light is a wave. The corpuscular hypothesis failed in 1803, when Thomas Young announced his **double-slit interference experiment** with light (see Figure 6.23), which firmly established light as a wave. In James Clerk Maxwell's theory of electromagnetism (completed by the year 1873), light is an electromagnetic wave. Maxwell's classical view of radiation as an electromagnetic wave is still valid today; however, it is unable to explain blackbody radiation and the photoelectric effect, where light acts as a beam of photons.

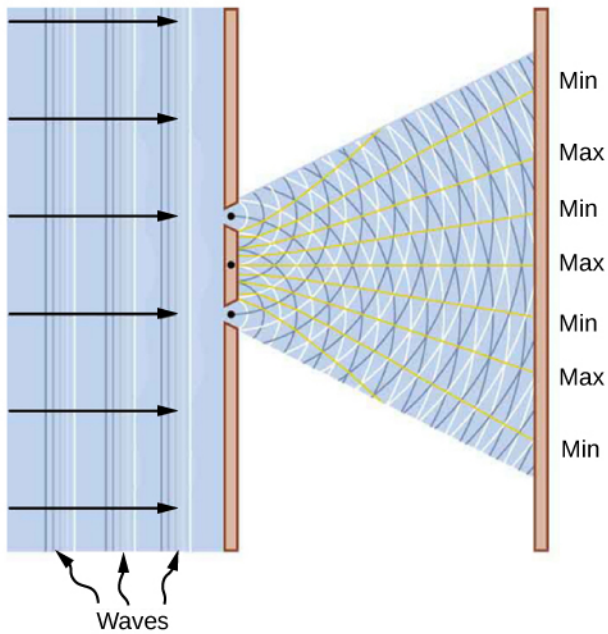


Figure 6.23 Young's double-slit experiment explains the interference of light by making an analogy with the interference of water waves. Two waves are generated at the positions of two slits in an opaque screen. The waves have the same wavelengths. They travel from their origins at the slits to the viewing screen placed to the right of the slits. The waves meet on the viewing screen. At the positions marked "Max" on the screen, the meeting waves are in-phase and the combined wave amplitude is enhanced. At positions marked "Min," the combined wave amplitude is zero. For light, this mechanism creates a bright-and-dark fringe pattern on the viewing screen.

A similar dichotomy existed in the interpretation of electricity. From Benjamin Franklin's observations of electricity in 1751 until J.J. Thomson's discovery of the electron in 1897, electric current was seen as a flow in a continuous electric medium. Within this theory of electric fluid, the present theory of electric circuits was developed, and electromagnetism and electromagnetic induction were discovered. Thomson's experiment showed that the unit of negative electric charge (an electron) can travel in a vacuum without any medium to carry the charge around, as in electric circuits. This discovery changed the way in which electricity is understood today and gave the electron its particle status. In Bohr's early quantum theory of the hydrogen atom, both the electron and the proton are particles of matter. Likewise, in the Compton scattering of X-rays on electrons, the electron is a particle. On the other hand, in electron-scattering experiments on crystalline structures, the electron behaves as a wave.

A skeptic may raise a question that perhaps an electron might always be nothing more than a particle, and that the diffraction images obtained in electron-scattering experiments might be explained within some macroscopic model of a crystal and a macroscopic model of electrons coming at it like a rain of ping-pong balls. As a matter of fact, to investigate this question, we do not need a complex model of a crystal but just a couple of simple slits in a screen that is opaque to electrons. In other words, to gather convincing evidence about the nature of an electron, we need to repeat the Young double-slit experiment with electrons. If the electron is a wave, we should observe the formation of interference patterns typical for waves, such as those described in **Figure 6.23**, even when electrons come through the slits one by one. However, if the electron is a not a wave but a particle, the interference fringes will not be formed.

The very first double-slit experiment with a beam of electrons, performed by Claus Jönsson in Germany in 1961, demonstrated that a beam of electrons indeed forms an interference pattern, which means that electrons collectively behave as a wave. The first double-slit experiments with *single* electrons passing through the slits one-by-one were performed by Giulio Pozzi in 1974 in Italy and by Akira Tonomura in 1989 in Japan. They show that interference fringes are formed gradually, even when electrons pass through the slits individually. This demonstrates conclusively that electron-diffraction images are formed because of the wave nature of electrons. The results seen in double-slit experiments with electrons are illustrated by the images of the interference pattern in **Figure 6.24**.

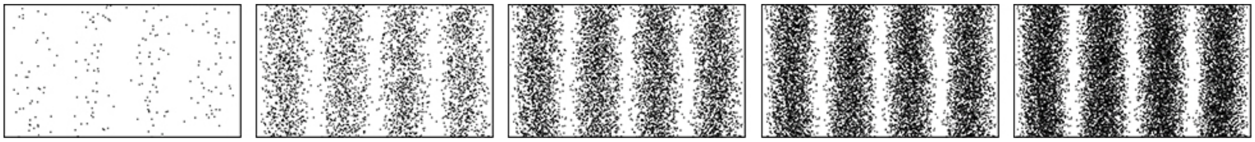


Figure 6.24 Computer-simulated interference fringes seen in the Young double-slit experiment with electrons. One pattern is gradually formed on the screen, regardless of whether the electrons come through the slits as a beam or individually one-by-one.

Example 6.15

Double-Slit Experiment with Electrons

In one experimental setup for studying interference patterns of electron waves, two slits are created in a gold-coated silicon membrane. Each slit is 62-nm wide and $4\text{-}\mu\text{m}$ long, and the separation between the slits is 272 nm.

The electron beam is created in an electron gun by heating a tungsten element and by accelerating the electrons across a 600-V potential. The beam is subsequently collimated using electromagnetic lenses, and the collimated beam of electrons is sent through the slits. Find the angular position of the first-order bright fringe on the viewing screen.

Strategy

Recall that the angular position θ of the nth order bright fringe that is formed in Young's two-slit interference pattern (discussed in a previous chapter) is related to the separation, d, between the slits and to the wavelength, $\lambda,$ of the incident light by the equation $d\sin\theta = n\lambda,$ where $n = 0, \pm 1, \pm 2, ...$ The separation is given and is equal to $d = 272\text{ nm}.$ For the first-order fringe, we take $n = 1.$ The only thing we now need is the wavelength of the incident electron wave.

Since the electron has been accelerated from rest across a potential difference of $\Delta V = 600\text{V},$ its kinetic energy is $K = e\Delta V = 600\text{ eV}.$ The rest-mass energy of the electron is $E_0 = 511\text{ keV}.$

We compute its de Broglie wavelength as that of a nonrelativistic electron because its kinetic energy K is much smaller than its rest energy $E_0,$ $K \ll E_0.$

Solution

The electron's wavelength is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m_e K}} = \frac{h}{\sqrt{2E_0/c^2 K}} = \frac{hc}{\sqrt{2E_0 K}} = \frac{1.241 \times 10^{-6}\text{eV} \cdot \text{m}}{\sqrt{2(511\text{ keV})(600\text{ eV})}} = 0.050\text{ nm}.$$

This λ is used to obtain the position of the first bright fringe:

$$\sin\theta = \frac{1 \cdot \lambda}{d} = \frac{0.050\text{ nm}}{272\text{ nm}} = 0.000184 \Rightarrow \theta = 0.010°.$$

Significance

Notice that this is also the angular resolution between two consecutive bright fringes up to about $n = 1000.$ For example, between the zero-order fringe and the first-order fringe, between the first-order fringe and the second-order fringe, and so on.

6.14 Check Your Understanding For the situation described in **Example 6.15**, find the angular position of the fifth-order bright fringe on the viewing screen.

The wave-particle dual nature of matter particles and of radiation is a declaration of our inability to describe physical reality within one unified classical theory because separately neither a classical particle approach nor a classical wave approach can fully explain the observed phenomena. This limitation of the classical approach was realized by the year 1928, and a foundation for a new statistical theory, called quantum mechanics, was put in place by Bohr, Edwin Schrödinger, Werner Heisenberg, and Paul Dirac. Quantum mechanics takes de Broglie's idea of matter waves to be the fundamental property of all particles and gives it a statistical interpretation. According to this interpretation, a wave that is associated with a particle carries information about the probable positions of the particle and about its other properties. A single particle is seen as a moving *wave packet* such as the one shown in **Figure 6.25**. We can intuitively sense from this example that if a particle is a wave packet, we will not be able to measure its exact position in the same sense as we cannot pinpoint a location of a wave packet in a vibrating guitar string. The uncertainty, Δx, in measuring the particle's position is connected to the uncertainty, Δp, in the simultaneous measuring of its linear momentum by Heisenberg's uncertainty principle:

$$\Delta x \Delta p \geq \frac{1}{2}\hbar. \quad (6.63)$$

Heisenberg's principle expresses the law of nature that, at the quantum level, our perception is limited. For example, if we know the exact position of a body (which means that $\Delta x = 0$ in **Equation 6.63**) at the same time we cannot know its momentum, because then the uncertainty in its momentum becomes infinite (because $\Delta p \geq 0.5\hbar/\Delta x$ in **Equation 6.63**). The **Heisenberg uncertainty principle** sets the limit on the precision of *simultaneous* measurements of position and momentum of a particle; it shows that the best precision we can obtain is when we have an equals sign ($=$) in **Equation 6.63**, and we cannot do better than that, even with the best instruments of the future. Heisenberg's principle is a consequence of the wave nature of particles.

Figure 6.25 In this graphic, a particle is shown as a wave packet and its position does not have an exact value.

We routinely use many electronic devices that exploit wave-particle duality without even realizing the sophistication of the physics underlying their operation. One example of a technology based on the particle properties of photons and electrons is a charge-coupled device, which is used for light detection in any instrumentation where high-quality digital data are required, such as in digital cameras or in medical sensors. An example in which the wave properties of electrons is exploited is an electron microscope.

In 1931, physicist Ernst Ruska—building on the idea that magnetic fields can direct an electron beam just as lenses can direct a beam of light in an optical microscope—developed the first prototype of the electron microscope. This development originated the field of **electron microscopy**. In the transmission electron microscope (TEM), shown in **Figure 6.26**, electrons are produced by a hot tungsten element and accelerated by a potential difference in an electron gun, which gives them up to 400 keV in kinetic energy. After leaving the electron gun, the electron beam is focused by electromagnetic lenses (a system of condensing lenses) and transmitted through a specimen sample to be viewed. The image of the sample is reconstructed from the transmitted electron beam. The magnified image may be viewed either directly on a fluorescent screen or indirectly by sending it, for example, to a digital camera or a computer monitor. The entire setup consisting of the electron gun, the lenses, the specimen, and the fluorescent screen are enclosed in a vacuum chamber to prevent the energy loss from the beam. Resolution of the TEM is limited only by spherical aberration (discussed in a previous chapter). Modern high-resolution models of a TEM can have resolving power greater than 0.5 Å and magnifications higher than 50 million times. For comparison, the best resolving power obtained with light microscopy is currently about 97 nm. A limitation of the TEM is that the samples must be about 100-nm thick and biological samples require a special preparation involving chemical "fixing" to stabilize them for ultrathin slicing.

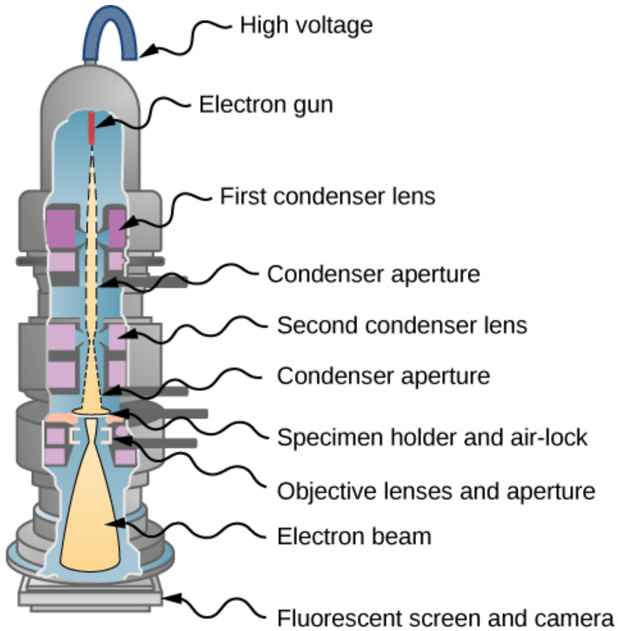


Figure 6.26 TEM: An electron beam produced by an electron gun is collimated by condenser lenses and passes through a specimen. The transmitted electrons are projected on a screen and the image is sent to a camera. (credit: modification of work by Dr. Graham Beards)

Such limitations do not appear in the scanning electron microscope (SEM), which was invented by Manfred von Ardenne in 1937. In an SEM, a typical energy of the electron beam is up to 40 keV and the beam is not transmitted through a sample but is scattered off its surface. Surface topography of the sample is reconstructed by analyzing back-scattered electrons, transmitted electrons, and the emitted radiation produced by electrons interacting with atoms in the sample. The resolving power of an SEM is better than 1 nm, and the magnification can be more than 250 times better than that obtained with a light microscope. The samples scanned by an SEM can be as large as several centimeters but they must be specially prepared, depending on electrical properties of the sample.

High magnifications of the TEM and SEM allow us to see individual molecules. High resolving powers of the TEM and SEM allow us to see fine details, such as those shown in the SEM micrograph of pollen at the beginning of this chapter (**Figure 6.1**).

Example 6.16

Resolving Power of an Electron Microscope

If a 1.0-pm electron beam of a TEM passes through a $2.0\text{-}\mu\text{m}$ circular opening, what is the angle between the two just-resolvable point sources for this microscope?

Solution

We can directly use a formula for the resolving power, $\Delta\theta,$ of a microscope (discussed in a previous chapter) when the wavelength of the incident radiation is $\lambda = 1.0\,\text{pm}$ and the diameter of the aperture is $D = 2.0\mu\text{m}$:

$$\Delta\theta = 1.22\frac{\lambda}{D} = 1.22\frac{1.0\,\text{pm}}{2.0\mu\text{m}} = 6.10\times 10^{-7}\,\text{rad} = 3.50\times 10^{-5}\,\text{degree}.$$

Significance

Note that if we used a conventional microscope with a 400-nm light, the resolving power would be only $14°,$ which means that all of the fine details in the image would be blurred.

6.15 Check Your Understanding Suppose that the diameter of the aperture in **Example 6.16** is halved. How does it affect the resolving power?

CHAPTER 6 REVIEW

KEY TERMS

absorber any object that absorbs radiation

absorption spectrum wavelengths of absorbed radiation by atoms and molecules

Balmer formula describes the emission spectrum of a hydrogen atom in the visible-light range

Balmer series spectral lines corresponding to electron transitions to/from the $n = 2$ state of the hydrogen atom, described by the Balmer formula

blackbody perfect absorber/emitter

blackbody radiation radiation emitted by a blackbody

Bohr radius of hydrogen radius of the first Bohr's orbit

Bohr's model of the hydrogen atom first quantum model to explain emission spectra of hydrogen

Brackett series spectral lines corresponding to electron transitions to/from the $n = 4$ state

Compton effect the change in wavelength when an X-ray is scattered by its interaction with some materials

Compton shift difference between the wavelengths of the incident X-ray and the scattered X-ray

Compton wavelength physical constant with the value $\lambda_c = 2.43$ pm

cut-off frequency frequency of incident light below which the photoelectric effect does not occur

cut-off wavelength wavelength of incident light that corresponds to cut-off frequency

Davisson–Germer experiment historically first electron-diffraction experiment that revealed electron waves

de Broglie wave matter wave associated with any object that has mass and momentum

de Broglie's hypothesis of matter waves particles of matter can behave like waves

double-slit interference experiment Young's double-slit experiment, which shows the interference of waves

electron microscopy microscopy that uses electron waves to "see" fine details of nano-size objects

emission spectrum wavelengths of emitted radiation by atoms and molecules

emitter any object that emits radiation

energy of a photon quantum of radiant energy, depends only on a photon's frequency

energy spectrum of hydrogen set of allowed discrete energies of an electron in a hydrogen atom

excited energy states of the H atom energy state other than the ground state

Fraunhofer lines dark absorption lines in the continuum solar emission spectrum

ground state energy of the hydrogen atom energy of an electron in the first Bohr orbit of the hydrogen atom

group velocity velocity of a wave, energy travels with the group velocity

Heisenberg uncertainty principle sets the limits on precision in simultaneous measurements of momentum and position of a particle

Humphreys series spectral lines corresponding to electron transitions to/from the $n = 6$ state

hydrogen-like atom ionized atom with one electron remaining and nucleus with charge $+Ze$

inelastic scattering scattering effect where kinetic energy is not conserved but the total energy is conserved

ionization energy energy needed to remove an electron from an atom

ionization limit of the hydrogen atom ionization energy needed to remove an electron from the first Bohr orbit

Lyman series spectral lines corresponding to electron transitions to/from the ground state

nuclear model of the atom heavy positively charged nucleus at the center is surrounded by electrons, proposed by Rutherford

Paschen series spectral lines corresponding to electron transitions to/from the $n = 3$ state

Pfund series spectral lines corresponding to electron transitions to/from the $n = 5$ state

photocurrent in a circuit, current that flows when a photoelectrode is illuminated

photoelectric effect emission of electrons from a metal surface exposed to electromagnetic radiation of the proper frequency

photoelectrode in a circuit, an electrode that emits photoelectrons

photoelectron electron emitted from a metal surface in the presence of incident radiation

photon particle of light

Planck's hypothesis of energy quanta energy exchanges between the radiation and the walls take place only in the form of discrete energy quanta

postulates of Bohr's model three assumptions that set a frame for Bohr's model

power intensity energy that passes through a unit surface per unit time

propagation vector vector with magnitude $2\pi/\lambda$ that has the direction of the photon's linear momentum

quantized energies discrete energies; not continuous

quantum number index that enumerates energy levels

quantum phenomenon in interaction with matter, photon transfers either all its energy or nothing

quantum state of a Planck's oscillator any mode of vibration of Planck's oscillator, enumerated by quantum number

reduced Planck's constant Planck's constant divided by 2π

Rutherford's gold foil experiment first experiment to demonstrate the existence of the atomic nucleus

Rydberg constant for hydrogen physical constant in the Balmer formula

Rydberg formula experimentally found positions of spectral lines of hydrogen atom

scattering angle angle between the direction of the scattered beam and the direction of the incident beam

Stefan–Boltzmann constant physical constant in Stefan's law

stopping potential in a circuit, potential difference that stops photocurrent

wave number magnitude of the propagation vector

wave quantum mechanics theory that explains the physics of atoms and subatomic particles

wave-particle duality particles can behave as waves and radiation can behave as particles

work function energy needed to detach photoelectron from the metal surface

α-particle doubly ionized helium atom

α-ray beam of α-particles (alpha-particles)

β-ray beam of electrons

γ-ray beam of highly energetic photons

KEY EQUATIONS

Wien's displacement law	$\lambda_{\text{max}} T = 2.898 \times 10^{-3}\,\text{m} \cdot \text{K}$
Stefan's law	$P(T) = \sigma A T^4$
Planck's constant	$h = 6.626 \times 10^{-34}\,\text{J} \cdot \text{s} = 4.136 \times 10^{-15}\,\text{eV} \cdot \text{s}$

Energy quantum of radiation	$\Delta E = hf$
Planck's blackbody radiation law	$I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1}$
Maximum kinetic energy of a photoelectron	$K_{\max} = e\Delta V_s$
Energy of a photon	$E_f = hf$
Energy balance for photoelectron	$K_{\max} = hf - \phi$
Cut-off frequency	$f_c = \frac{\phi}{h}$
Relativistic invariant energy equation	$E^2 = p^2 c^2 + m_0^2 c^4$
Energy-momentum relation for photon	$p_f = \frac{E_f}{c}$
Energy of a photon	$E_f = hf = \frac{hc}{\lambda}$
Magnitude of photon's momentum	$p_f = \frac{h}{\lambda}$
Photon's linear momentum vector	$\vec{p}_f = \hbar \vec{k}$
The Compton wavelength of an electron	$\lambda_c = \frac{h}{m_0 c} = 0.00243 \text{ nm}$
The Compton shift	$\Delta\lambda = \lambda_c(1 - \cos\theta)$
The Balmer formula	$\frac{1}{\lambda} = R_{\text{H}}\left(\frac{1}{2^2} - \frac{1}{n^2}\right)$
The Rydberg formula	$\frac{1}{\lambda} = R_{\text{H}}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right), n_i = n_f + 1, n_f + 2, \ldots$
Bohr's first quantization condition	$L_n = n\hbar, n = 1, 2, \ldots$
Bohr's second quantization condition	$hf = \lvert E_n - E_m \rvert$
Bohr's radius of hydrogen	$a_0 = 4\pi\varepsilon_0 \frac{\hbar^2}{m_e e^2} = 0.529\text{Å}$
Bohr's radius of the *n*th orbit	$r_n = a_0 n^2$
Ground-state energy value, ionization limit	$E_0 = \frac{1}{8\varepsilon_0^2} \frac{m_e e^4}{h^2} = 13.6 \text{ eV}$
Electron's energy in the *n*th orbit	$E_n = -E_0 \frac{1}{n^2}$
Ground state energy of hydrogen	$E_1 = -E_0 = -13.6 \text{ eV}$
The *n*th orbit of hydrogen-like ion	$r_n = \frac{a_0}{Z} n^2$

The *n*th energy of hydrogen-like ion	$E_n = -Z^2 E_0 \frac{1}{n^2}$
Energy of a matter wave	$E = hf$
The de Broglie wavelength	$\lambda = \frac{h}{p}$
The frequency-wavelength relation for matter waves	$\lambda f = \frac{c}{\beta}$
Heisenberg's uncertainty principle	$\Delta x \Delta p \geq \frac{1}{2}\hbar$

SUMMARY

6.1 Blackbody Radiation

- All bodies radiate energy. The amount of radiation a body emits depends on its temperature. The experimental Wien's displacement law states that the hotter the body, the shorter the wavelength corresponding to the emission peak in the radiation curve. The experimental Stefan's law states that the total power of radiation emitted across the entire spectrum of wavelengths at a given temperature is proportional to the fourth power of the Kelvin temperature of the radiating body.
- Absorption and emission of radiation are studied within the model of a blackbody. In the classical approach, the exchange of energy between radiation and cavity walls is continuous. The classical approach does not explain the blackbody radiation curve.
- To explain the blackbody radiation curve, Planck assumed that the exchange of energy between radiation and cavity walls takes place only in discrete quanta of energy. Planck's hypothesis of energy quanta led to the theoretical Planck's radiation law, which agrees with the experimental blackbody radiation curve; it also explains Wien's and Stefan's laws.

6.2 Photoelectric Effect

- The photoelectric effect occurs when photoelectrons are ejected from a metal surface in response to monochromatic radiation incident on the surface. It has three characteristics: (1) it is instantaneous, (2) it occurs only when the radiation is above a cut-off frequency, and (3) kinetic energies of photoelectrons at the surface do not depend of the intensity of radiation. The photoelectric effect cannot be explained by classical theory.
- We can explain the photoelectric effect by assuming that radiation consists of photons (particles of light). Each photon carries a quantum of energy. The energy of a photon depends only on its frequency, which is the frequency of the radiation. At the surface, the entire energy of a photon is transferred to one photoelectron.
- The maximum kinetic energy of a photoelectron at the metal surface is the difference between the energy of the incident photon and the work function of the metal. The work function is the binding energy of electrons to the metal surface. Each metal has its own characteristic work function.

6.3 The Compton Effect

- In the Compton effect, X-rays scattered off some materials have different wavelengths than the wavelength of the incident X-rays. This phenomenon does not have a classical explanation.
- The Compton effect is explained by assuming that radiation consists of photons that collide with weakly bound electrons in the target material. Both electron and photon are treated as relativistic particles. Conservation laws of the total energy and of momentum are obeyed in collisions.
- Treating the photon as a particle with momentum that can be transferred to an electron leads to a theoretical Compton shift that agrees with the wavelength shift measured in the experiment. This provides evidence that radiation consists of photons.
- Compton scattering is an inelastic scattering, in which scattered radiation has a longer wavelength than that of incident radiation.

6.4 Bohr’s Model of the Hydrogen Atom

- Positions of absorption and emission lines in the spectrum of atomic hydrogen are given by the experimental Rydberg formula. Classical physics cannot explain the spectrum of atomic hydrogen.
- The Bohr model of hydrogen was the first model of atomic structure to correctly explain the radiation spectra of atomic hydrogen. It was preceded by the Rutherford nuclear model of the atom. In Rutherford’s model, an atom consists of a positively charged point-like nucleus that contains almost the entire mass of the atom and of negative electrons that are located far away from the nucleus.
- Bohr’s model of the hydrogen atom is based on three postulates: (1) an electron moves around the nucleus in a circular orbit, (2) an electron’s angular momentum in the orbit is quantized, and (3) the change in an electron’s energy as it makes a quantum jump from one orbit to another is always accompanied by the emission or absorption of a photon. Bohr’s model is semi-classical because it combines the classical concept of electron orbit (postulate 1) with the new concept of quantization (postulates 2 and 3).
- Bohr’s model of the hydrogen atom explains the emission and absorption spectra of atomic hydrogen and hydrogen-like ions with low atomic numbers. It was the first model to introduce the concept of a quantum number to describe atomic states and to postulate quantization of electron orbits in the atom. Bohr’s model is an important step in the development of quantum mechanics, which deals with many-electron atoms.

6.5 De Broglie’s Matter Waves

- De Broglie’s hypothesis of matter waves postulates that any particle of matter that has linear momentum is also a wave. The wavelength of a matter wave associated with a particle is inversely proportional to the magnitude of the particle’s linear momentum. The speed of the matter wave is the speed of the particle.
- De Broglie’s concept of the electron matter wave provides a rationale for the quantization of the electron’s angular momentum in Bohr’s model of the hydrogen atom.
- In the Davisson–Germer experiment, electrons are scattered off a crystalline nickel surface. Diffraction patterns of electron matter waves are observed. They are the evidence for the existence of matter waves. Matter waves are observed in diffraction experiments with various particles.

6.6 Wave-Particle Duality

- Wave-particle duality exists in nature: Under some experimental conditions, a particle acts as a particle; under other experimental conditions, a particle acts as a wave. Conversely, under some physical circumstances, electromagnetic radiation acts as a wave, and under other physical circumstances, radiation acts as a beam of photons.
- Modern-era double-slit experiments with electrons demonstrated conclusively that electron-diffraction images are formed because of the wave nature of electrons.
- The wave-particle dual nature of particles and of radiation has no classical explanation.
- Quantum theory takes the wave property to be the fundamental property of all particles. A particle is seen as a moving wave packet. The wave nature of particles imposes a limitation on the simultaneous measurement of the particle’s position and momentum. Heisenberg’s uncertainty principle sets the limits on precision in such simultaneous measurements.
- Wave-particle duality is exploited in many devices, such as charge-couple devices (used in digital cameras) or in the electron microscopy of the scanning electron microscope (SEM) and the transmission electron microscope (TEM).

CONCEPTUAL QUESTIONS

6.1 Blackbody Radiation

1. Which surface has a higher temperature – the surface of a yellow star or that of a red star?

2. Describe what you would see when looking at a body whose temperature is increased from 1000 K to 1,000,000 K.

3. Explain the color changes in a hot body as its temperature is increased.

4. Speculate as to why UV light causes sunburn, whereas visible light does not.

5. Two cavity radiators are constructed with walls made of different metals. At the same temperature, how would their radiation spectra differ?

6. Discuss why some bodies appear black, other bodies appear red, and still other bodies appear white.

7. If everything radiates electromagnetic energy, why can we not see objects at room temperature in a dark room?

8. How much does the power radiated by a blackbody increase when its temperature (in K) is tripled?

6.2 Photoelectric Effect

9. For the same monochromatic light source, would the photoelectric effect occur for all metals?

10. In the interpretation of the photoelectric effect, how is it known that an electron does not absorb more than one photon?

11. Explain how you can determine the work function from a plot of the stopping potential versus the frequency of the incident radiation in a photoelectric effect experiment. Can you determine the value of Planck's constant from this plot?

12. Suppose that in the photoelectric-effect experiment we make a plot of the detected current versus the applied potential difference. What information do we obtain from such a plot? Can we determine from it the value of Planck's constant? Can we determine the work function of the metal?

13. Speculate how increasing the temperature of a photoelectrode affects the outcomes of the photoelectric effect experiment.

14. Which aspects of the photoelectric effect cannot be explained by classical physics?

15. Is the photoelectric effect a consequence of the wave character of radiation or is it a consequence of the particle character of radiation? Explain briefly.

16. The metals sodium, iron, and molybdenum have work functions 2.5 eV, 3.9 eV, and 4.2 eV, respectively. Which of these metals will emit photoelectrons when illuminated with 400 nm light?

6.3 The Compton Effect

17. Discuss any similarities and differences between the photoelectric and the Compton effects.

18. Which has a greater momentum: an UV photon or an IR photon?

19. Does changing the intensity of a monochromatic light beam affect the momentum of the individual photons in the beam? Does such a change affect the net momentum of the beam?

20. Can the Compton effect occur with visible light? If so, will it be detectable?

21. Is it possible in the Compton experiment to observe scattered X-rays that have a shorter wavelength than the incident X-ray radiation?

22. Show that the Compton wavelength has the dimension of length.

23. At what scattering angle is the wavelength shift in the Compton effect equal to the Compton wavelength?

6.4 Bohr's Model of the Hydrogen Atom

24. Explain why the patterns of bright emission spectral lines have an identical spectral position to the pattern of dark absorption spectral lines for a given gaseous element.

25. Do the various spectral lines of the hydrogen atom overlap?

26. The Balmer series for hydrogen was discovered before either the Lyman or the Paschen series. Why?

27. When the absorption spectrum of hydrogen at room temperature is analyzed, absorption lines for the Lyman series are found, but none are found for the Balmer series. What does this tell us about the energy state of most hydrogen atoms at room temperature?

28. Hydrogen accounts for about 75% by mass of the matter at the surfaces of most stars. However, the absorption lines of hydrogen are strongest (of highest intensity) in the spectra of stars with a surface temperature of about 9000 K. They are weaker in the sun spectrum and are essentially nonexistent in very hot (temperatures above 25,000 K) or rather cool (temperatures below 3500 K) stars. Speculate as to why surface temperature affects the hydrogen absorption lines that we observe.

29. Discuss the similarities and differences between Thomson's model of the hydrogen atom and Bohr's model of the hydrogen atom.

30. Discuss the way in which Thomson's model is nonphysical. Support your argument with experimental evidence.

31. If, in a hydrogen atom, an electron moves to an orbit with a larger radius, does the energy of the hydrogen atom increase or decrease?

32. How is the energy conserved when an atom makes a transition from a higher to a lower energy state?

33. Suppose an electron in a hydrogen atom makes a transition from the (*n+1*)th orbit to the *n*th orbit. Is the wavelength of the emitted photon longer for larger values of *n*, or for smaller values of *n*?

34. Discuss why the allowed energies of the hydrogen atom are negative.

35. Can a hydrogen atom absorb a photon whose energy is greater than 13.6 eV?

36. Why can you see through glass but not through wood?

37. Do gravitational forces have a significant effect on atomic energy levels?

38. Show that Planck's constant has the dimensions of angular momentum.

6.5 De Broglie's Matter Waves

39. Which type of radiation is most suitable for the observation of diffraction patterns on crystalline solids; radio waves, visible light, or X-rays? Explain.

40. Speculate as to how the diffraction patterns of a typical crystal would be affected if γ-rays were used instead of X-rays.

41. If an electron and a proton are traveling at the same speed, which one has the shorter de Broglie wavelength?

42. If a particle is accelerating, how does this affect its de Broglie wavelength?

43. Why is the wave-like nature of matter not observed every day for macroscopic objects?

44. What is the wavelength of a neutron at rest? Explain.

45. Why does the setup of Davisson–Germer experiment need to be enclosed in a vacuum chamber? Discuss what result you expect when the chamber is not evacuated.

6.6 Wave-Particle Duality

46. Give an example of an experiment in which light behaves as waves. Give an example of an experiment in which light behaves as a stream of photons.

47. Discuss: How does the interference of water waves differ from the interference of electrons? How are they analogous?

48. Give at least one argument in support of the matter-wave hypothesis.

49. Give at least one argument in support of the particle-nature of radiation.

50. Explain the importance of the Young double-slit experiment.

51. Does the Heisenberg uncertainty principle allow a particle to be at rest in a designated region in space?

52. Can the de Broglie wavelength of a particle be known exactly?

53. Do the photons of red light produce better resolution in a microscope than blue light photons? Explain.

54. Discuss the main difference between an SEM and a TEM.

PROBLEMS

6.1 Blackbody Radiation

55. A 200-W heater emits a 1.5-μm radiation. (a) What value of the energy quantum does it emit? (b) Assuming that the specific heat of a 4.0-kg body is $0.83\,\text{kcal/kg}\cdot\text{K}$, how many of these photons must be absorbed by the body to increase its temperature by 2 K? (c) How long does the heating process in (b) take, assuming that all radiation emitted by the heater gets absorbed by the body?

56. A 900-W microwave generator in an oven generates energy quanta of frequency 2560 MHz. (a) How many energy quanta does it emit per second? (b) How many energy quanta must be absorbed by a pasta dish placed in the radiation cavity to increase its temperature by 45.0 K? Assume that the dish has a mass of 0.5 kg and that its specific heat is $0.9\,\text{kcal/kg}\cdot\text{K}$. (c) Assume that all energy quanta emitted by the generator are absorbed by the pasta dish. How long must we wait until the dish in (b) is ready?

57. (a) For what temperature is the peak of blackbody radiation spectrum at 400 nm? (b) If the temperature of a blackbody is 800 K, at what wavelength does it radiate the most energy?

58. The tungsten elements of incandescent light bulbs operate at 3200 K. At what frequency does the filament radiate maximum energy?

59. Interstellar space is filled with radiation of wavelength $970\mu\text{m}$. This radiation is considered to be a remnant of the "big bang." What is the corresponding blackbody temperature of this radiation?

60. The radiant energy from the sun reaches its maximum at a wavelength of about 500.0 nm. What is the approximate temperature of the sun's surface?

6.2 Photoelectric Effect

61. A photon has energy 20 keV. What are its frequency and wavelength?

62. The wavelengths of visible light range from approximately 400 to 750 nm. What is the corresponding range of photon energies for visible light?

63. What is the longest wavelength of radiation that can eject a photoelectron from silver? Is it in the visible range?

64. What is the longest wavelength of radiation that can eject a photoelectron from potassium, given the work function of potassium 2.24 eV? Is it in the visible range?

65. Estimate the binding energy of electrons in magnesium, given that the wavelength of 337 nm is the longest wavelength that a photon may have to eject a photoelectron from magnesium photoelectrode.

66. The work function for potassium is 2.26 eV. What is the cutoff frequency when this metal is used as photoelectrode? What is the stopping potential when for the emitted electrons when this photoelectrode is exposed to radiation of frequency 1200 THz?

67. Estimate the work function of aluminum, given that the wavelength of 304 nm is the longest wavelength that a photon may have to eject a photoelectron from aluminum photoelectrode.

68. What is the maximum kinetic energy of photoelectrons ejected from sodium by the incident radiation of wavelength 450 nm?

69. A 120-nm UV radiation illuminates a gold-plated electrode. What is the maximum kinetic energy of the ejected photoelectrons?

70. A 400-nm violet light ejects photoelectrons with a maximum kinetic energy of 0.860 eV from sodium photoelectrode. What is the work function of sodium?

71. A 600-nm light falls on a photoelectric surface and electrons with the maximum kinetic energy of 0.17 eV are emitted. Determine (a) the work function and (b) the cutoff frequency of the surface. (c) What is the stopping potential when the surface is illuminated with light of wavelength 400 nm?

72. The cutoff wavelength for the emission of photoelectrons from a particular surface is 500 nm. Find the maximum kinetic energy of the ejected photoelectrons when the surface is illuminated with light of wavelength 600 nm.

73. Find the wavelength of radiation that can eject 2.00-eV electrons from calcium electrode. The work function for calcium is 2.71 eV. In what range is this radiation?

74. Find the wavelength of radiation that can eject 0.10-eV electrons from potassium electrode. The work function for potassium is 2.24 eV. In what range is this radiation?

75. Find the maximum velocity of photoelectrons ejected by an 80-nm radiation, if the work function of photoelectrode is 4.73 eV.

6.3 The Compton Effect

76. What is the momentum of a 589-nm yellow photon?

77. What is the momentum of a 4-cm microwave photon?

78. In a beam of white light (wavelengths from 400 to 750 nm), what range of momentum can the photons have?

79. What is the energy of a photon whose momentum is $3.0 \times 10^{-24}\,\text{kg} \cdot \text{m/s}$?

80. What is the wavelength of (a) a 12-keV X-ray photon; (b) a 2.0-MeV γ -ray photon?

81. Find the momentum and energy of a 1.0-Å photon.

82. Find the wavelength and energy of a photon with momentum $5.00 \times 10^{-29}\,\text{kg} \cdot \text{m/s}$.

83. A γ-ray photon has a momentum of $8.00 \times 10^{-21}\,\text{kg}\cdot\text{m/s}$. Find its wavelength and energy.

84. (a) Calculate the momentum of a $2.5\text{-}\mu\text{m}$ photon. (b) Find the velocity of an electron with the same momentum. (c) What is the kinetic energy of the electron, and how does it compare to that of the photon?

85. Show that $p = h/\lambda$ and $E_f = hf$ are consistent with the relativistic formula $E^2 = p^2c^2 + m_0^2c^2$.

86. Show that the energy E in eV of a photon is given by $E = 1.241 \times 10^{-6}\,\text{eV}\cdot\text{m}/\lambda$, where λ is its wavelength in meters.

87. For collisions with free electrons, compare the Compton shift of a photon scattered as an angle of $30°$ to that of a photon scattered at $45°$.

88. X-rays of wavelength 12.5 pm are scattered from a block of carbon. What are the wavelengths of photons scattered at (a) $30°$; (b) $90°$; and, (c) $180°$?

6.4 Bohr's Model of the Hydrogen Atom

89. Calculate the wavelength of the first line in the Lyman series and show that this line lies in the ultraviolet part of the spectrum.

90. Calculate the wavelength of the fifth line in the Lyman series and show that this line lies in the ultraviolet part of the spectrum.

91. Calculate the energy changes corresponding to the transitions of the hydrogen atom: (a) from $n = 3$ to $n = 4$; (b) from $n = 2$ to $n = 1$; and (c) from $n = 3$ to $n = \infty$.

92. Determine the wavelength of the third Balmer line (transition from $n = 5$ to $n = 2$).

93. What is the frequency of the photon absorbed when the hydrogen atom makes the transition from the ground state to the $n = 4$ state?

94. When a hydrogen atom is in its ground state, what are the shortest and longest wavelengths of the photons it can absorb without being ionized?

95. When a hydrogen atom is in its third excided state, what are the shortest and longest wavelengths of the photons it can emit?

96. What is the longest wavelength that light can have if it is to be capable of ionizing the hydrogen atom in its ground state?

97. For an electron in a hydrogen atom in the $n = 2$ state, compute: (a) the angular momentum; (b) the kinetic energy; (c) the potential energy; and (d) the total energy.

98. Find the ionization energy of a hydrogen atom in the fourth energy state.

99. It has been measured that it required 0.850 eV to remove an electron from the hydrogen atom. In what state was the atom before the ionization happened?

100. What is the radius of a hydrogen atom when the electron is in the first excited state?

101. Find the shortest wavelength in the Balmer series. In what part of the spectrum does this line lie?

102. Show that the entire Paschen series lies in the infrared part of the spectrum.

103. Do the Balmer series and the Lyman series overlap? Why? Why not? (Hint: calculate the shortest Balmer line and the longest Lyman line.)

104. (a) Which line in the Balmer series is the first one in the UV part of the spectrum? (b) How many Balmer lines lie in the visible part of the spectrum? (c) How many Balmer lines lie in the UV?

105. A $4.653\text{-}\mu\text{m}$ emission line of atomic hydrogen corresponds to transition between the states $n_f = 5$ and n_i. Find n_i.

6.5 De Broglie's Matter Waves

106. At what velocity will an electron have a wavelength of 1.00 m?

107. What is the de Broglie wavelength of an electron travelling at a speed of $5.0 \times 10^6\,\text{m/s}$?

108. What is the de Broglie wavelength of an electron that is accelerated from rest through a potential difference of 20 keV?

109. What is the de Broglie wavelength of a proton whose kinetic energy is 2.0 MeV? 10.0 MeV?

110. What is the de Broglie wavelength of a 10-kg football player running at a speed of 8.0 m/s?

111. (a) What is the energy of an electron whose de Broglie wavelength is that of a photon of yellow light with wavelength 590 nm? (b) What is the de Broglie wavelength of an electron whose energy is that of the photon of yellow light?

112. The de Broglie wavelength of a neutron is 0.01 nm. What is the speed and energy of this neutron?

113. What is the wavelength of an electron that is moving at a 3% of the speed of light?

114. At what velocity does a proton have a 6.0-fm wavelength (about the size of a nucleus)? Give your answer in units of *c*.

115. What is the velocity of a 0.400-kg billiard ball if its wavelength is 7.50 fm?

116. Find the wavelength of a proton that is moving at 1.00% of the speed of light (when $\beta = 0.01$).

6.6 Wave-Particle Duality

117. An AM radio transmitter radiates 500 kW at a frequency of 760 kHz. How many photons per second does the emitter emit?

118. Find the Lorentz factor γ and de Broglie's wavelength for a 50-GeV electron in a particle accelerator.

119. Find the Lorentz factor γ and de Broglie's wavelength for a 1.0-TeV proton in a particle accelerator.

120. What is the kinetic energy of a 0.01-nm electron in a TEM?

121. If electron is to be diffracted significantly by a crystal, its wavelength must be about equal to the spacing, *d*, of crystalline planes. Assuming $d = 0.250\,\text{nm}$, estimate the potential difference through which an electron must be accelerated from rest if it is to be diffracted by these planes.

122. X-rays form ionizing radiation that is dangerous to living tissue and undetectable to the human eye. Suppose that a student researcher working in an X-ray diffraction laboratory is accidentally exposed to a fatal dose of radiation. Calculate the temperature increase of the researcher under the following conditions: the energy of X-ray photons is 200 keV and the researcher absorbs 4×10^{13} photons per each kilogram of body weight during the exposure. Assume that the specific heat of the student's body is $0.83\text{kcal/kg} \cdot \text{K}$.

123. Solar wind (radiation) that is incident on the top of Earth's atmosphere has an average intensity of 1.3kW/m^2. Suppose that you are building a solar sail that is to propel a small toy spaceship with a mass of 0.1 kg in the space between the International Space Station and the moon. The sail is made from a very light material, which perfectly reflects the incident radiation. To assess whether such a project is feasible, answer the following questions, assuming that radiation photons are incident only in normal direction to the sail reflecting surface. (a) What is the radiation pressure (force per m^2) of the radiation falling on the mirror-like sail? (b) Given the radiation pressure computed in (a), what will be the acceleration of the spaceship when the sail has of an area of 10.0m^2 ? (c) Given the acceleration estimate in (b), how fast will the spaceship be moving after 24 hours when it starts from rest?

124. Treat the human body as a blackbody and determine the percentage increase in the total power of its radiation when its temperature increases from 98.6 ° F to 103 ° F.

125. Show that Wien's displacement law results from Planck's radiation law. (*Hint:* substitute $x = hc/\lambda kT$ and write Planck's law in the form $I(x, T) = Ax^5/(e^x - 1)$, where $A = 2\pi(kT)^5/(h^4c^3)$. Now, for fixed *T*, find the position of the maximum in *I*(*x*,*T*) by solving for *x* in the equation $dI(x, T)/dx = 0.$)

126. Show that Stefan's law results from Planck's radiation law. *Hint:* To compute the total power of blackbody radiation emitted across the entire spectrum of wavelengths at a given temperature, integrate Planck's law over the entire spectrum $P(T) = \int_0^\infty I(\lambda, T)d\lambda$. Use the substitution $x = hc/\lambda kT$ and the tabulated value of the integral $\int_0^\infty dx x^3/(e^x - 1) = \pi^4/15$.

ADDITIONAL PROBLEMS

127. Determine the power intensity of radiation per unit wavelength emitted at a wavelength of 500.0 nm by a blackbody at a temperature of 10,000 K.

128. The HCl molecule oscillates at a frequency of 87.0 THz. What is the difference (in eV) between its adjacent energy levels?

129. A quantum mechanical oscillator vibrates at a frequency of 250.0 THz. What is the minimum energy of radiation it can emit?

130. In about 5 billion years, the sun will evolve to a red giant. Assume that its surface temperature will decrease to about half its present value of 6000 K, while its present radius of $7.0 \times 10^8\,\text{m}$ will increase to $1.5 \times 10^{11}\,\text{m}$ (which is the current Earth-sun distance). Calculate the ratio of the total power emitted by the sun in its red giant stage to its present power.

131. A sodium lamp emits 2.0 W of radiant energy, most of which has a wavelength of about 589 nm. Estimate the number of photons emitted per second by the lamp.

132. Photoelectrons are ejected from a photoelectrode and are detected at a distance of 2.50 cm away from the photoelectrode. The work function of the photoelectrode is 2.71 eV and the incident radiation has a wavelength of 420 nm. How long does it take a photoelectron to travel to the detector?

133. If the work function of a metal is 3.2 eV, what is the maximum wavelength that a photon can have to eject a photoelectron from this metal surface?

134. The work function of a photoelectric surface is 2.00 eV. What is the maximum speed of the photoelectrons emitted from this surface when a 450-nm light falls on it?

135. A 400-nm laser beam is projected onto a calcium electrode. The power of the laser beam is 2.00 mW and the work function of calcium is 2.31 eV. (a) How many photoelectrons per second are ejected? (b) What net power is carried away by photoelectrons?

136. (a) Calculate the number of photoelectrons per second that are ejected from a 1.00-mm^2 area of sodium metal by a 500-nm radiation with intensity 1.30kW/m^2 (the intensity of sunlight above Earth's atmosphere). (b) Given the work function of the metal as 2.28 eV, what power is carried away by these photoelectrons?

137. A laser with a power output of 2.00 mW at a 400-nm wavelength is used to project a beam of light onto a calcium photoelectrode. (a) How many photoelectrons leave the calcium surface per second? (b) What power is carried away by ejected photoelectrons, given that the work function of calcium is 2.31 eV? (c) Calculate the photocurrent. (d) If the photoelectrode suddenly becomes electrically insulated and the setup of two electrodes in the circuit suddenly starts to act like a 2.00-pF capacitor, how long will current flow before the capacitor voltage stops it?

138. The work function for barium is 2.48 eV. Find the maximum kinetic energy of the ejected photoelectrons when the barium surface is illuminated with: (a) radiation emitted by a 100-kW radio station broadcasting at 800 kHz; (b) a 633-nm laser light emitted from a powerful He-Ne laser; and (c) a 434-nm blue light emitted by a small hydrogen gas discharge tube.

139. (a) Calculate the wavelength of a photon that has the same momentum as a proton moving with 1% of the speed of light in a vacuum. (b) What is the energy of this photon in MeV? (c) What is the kinetic energy of the proton in MeV?

140. (a) Find the momentum of a 100-keV X-ray photon. (b) Find the velocity of a neutron with the same momentum. (c) What is the neutron's kinetic energy in eV?

141. The momentum of light, as it is for particles, is exactly reversed when a photon is reflected straight back from a mirror, assuming negligible recoil of the mirror. The change in momentum is twice the photon's incident momentum, as it is for the particles. Suppose that a beam of light has an intensity 1.0kW/m^2 and falls on a -2.0-m^2 area of a mirror and reflects from it. (a) Calculate the energy reflected in 1.00 s. (b) What is the momentum imparted to the mirror? (c) Use Newton's second law to find the force on the mirror. (d) Does the assumption of no-recoil for the mirror seem reasonable?

142. A photon of energy 5.0 keV collides with a stationary electron and is scattered at an angle of $60°$. What is the energy acquired by the electron in the collision?

143. A 0.75-nm photon is scattered by a stationary electron. The speed of the electron's recoil is $1.5 \times 10^6\,\text{m/s}$. (a) Find the wavelength shift of the photon. (b) Find the scattering angle of the photon.

144. Find the maximum change in X-ray wavelength that can occur due to Compton scattering. Does this change depend on the wavelength of the incident beam?

145. A photon of wavelength 700 nm is incident on a hydrogen atom. When this photon is absorbed, the atom becomes ionized. What is the lowest possible orbit that the electron could have occupied before being ionized?

146. What is the maximum kinetic energy of an electron such that a collision between the electron and a stationary hydrogen atom in its ground state is definitely elastic?

147. Singly ionized atomic helium He^{+1} is a hydrogen-like ion. (a) What is its ground-state radius? (b) Calculate the energies of its four lowest energy states. (c) Repeat the calculations for the Li^{2+} ion.

148. A triply ionized atom of beryllium Be^{3+} is a hydrogen-like ion. When Be^{3+} is in one of its excited states, its radius in this *n*th state is exactly the same as the radius of the first Bohr orbit of hydrogen. Find *n* and compute the ionization energy for this state of Be^{3+}.

149. In extreme-temperature environments, such as those existing in a solar corona, atoms may be ionized by undergoing collisions with other atoms. One example of such ionization in the solar corona is the presence of C^{5+} ions, detected in the Fraunhofer spectrum. (a) By what factor do the energies of the C^{5+} ion scale compare to the energy spectrum of a hydrogen atom? (b) What is the wavelength of the first line in the Paschen series of C^{5+} ? (c) In what part of the spectrum are these lines located?

150. (a) Calculate the ionization energy for He^{+}. (b) What is the minimum frequency of a photon capable of ionizing He^{+} ?

151. Experiments are performed with ultracold neutrons having velocities as small as 1.00 m/s. Find the wavelength of such an ultracold neutron and its kinetic energy.

152. Find the velocity and kinetic energy of a 6.0-fm neutron. (Rest mass energy of neutron is $E_0 = 940\,\text{MeV}$.)

153. The spacing between crystalline planes in the NaCl crystal is 0.281 nm, as determined by X-ray diffraction with X-rays of wavelength 0.170 nm. What is the energy of neutrons in the neutron beam that produces diffraction peaks at the same locations as the peaks obtained with the X-rays?

154. What is the wavelength of an electron accelerated from rest in a 30.0-kV potential difference?

155. Calculate the velocity of a $1.0\text{-}\mu\text{m}$ electron and a potential difference used to accelerate it from rest to this velocity.

156. In a supercollider at CERN, protons are accelerated to velocities of 0.25*c*. What are their wavelengths at this speed? What are their kinetic energies? If a beam of protons were to gain its kinetic energy in only one pass through a potential difference, how high would this potential difference have to be? (Rest mass energy of a proton is $E_0 = 938\,\text{MeV}$).

157. Find the de Broglie wavelength of an electron accelerated from rest in an X-ray tube in the potential difference of 100 keV. (Rest mass energy of an electron is $E_0 = 511\,\text{keV}$.)

158. The cutoff wavelength for the emission of photoelectrons from a particular surface is 500 nm. Find the maximum kinetic energy of the ejected photoelectrons when the surface is illuminated with light of wavelength 450 nm.

159. Compare the wavelength shift of a photon scattered by a free electron to that of a photon scattered at the same angle by a free proton.

160. The spectrometer used to measure the wavelengths of the scattered X-rays in the Compton experiment is accurate to $5.0 \times 10^{-4}\,\text{nm}$. What is the minimum scattering angle for which the X-rays interacting with the free electrons can be distinguished from those interacting with the atoms?

161. Consider a hydrogen-like ion where an electron is orbiting a nucleus that has charge $q = +Ze$. Derive the formulas for the energy E_n of the electron in *n*th orbit and the orbital radius r_n.

162. Assume that a hydrogen atom exists in the $n = 2$ excited state for $10^{-8}\,\text{s}$ before decaying to the ground state. How many times does the electron orbit the proton nucleus during this time? How long does it take Earth to orbit the sun this many times?

163. An atom can be formed when a negative muon is captured by a proton. The muon has the same charge as the electron and a mass 207 times that of the electron. Calculate the frequency of the photon emitted when this atom makes the transition from $n = 2$ to the $n = 1$ state. Assume that the muon is orbiting a stationary proton.

7 | QUANTUM MECHANICS

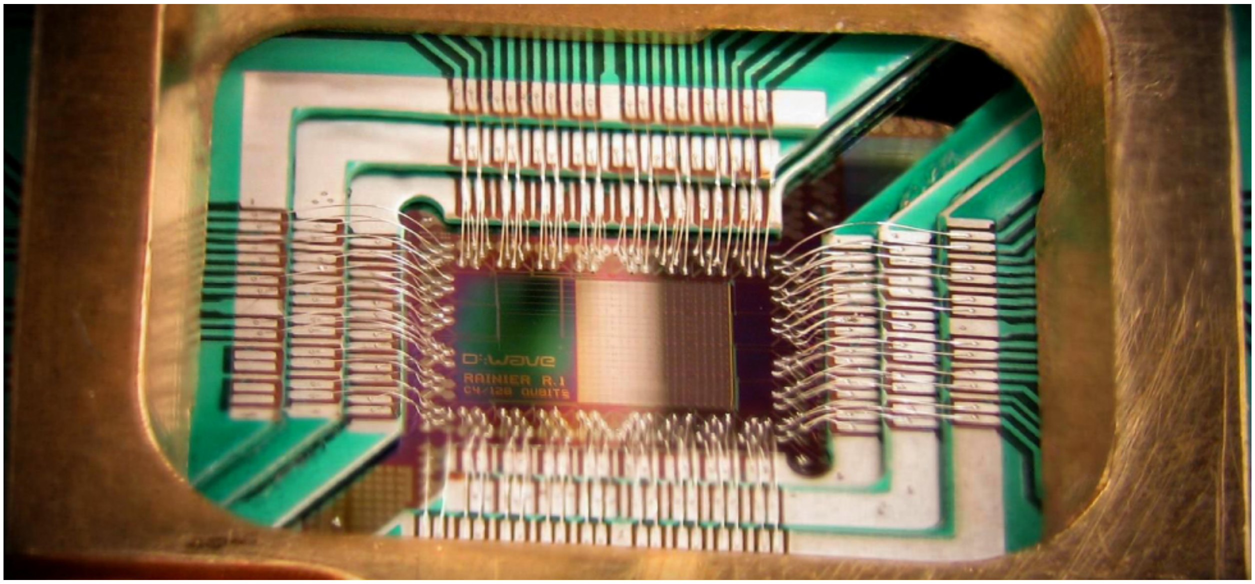


Figure 7.1 A D-wave qubit processor: The brain of a quantum computer that encodes information in quantum bits to perform complex calculations. (credit: modification of work by D-Wave Systems, Inc.)

Chapter Outline

Introduction

Quantum mechanics is a powerful framework for understanding the motions and interactions of particles at small scales, such as atoms and molecules. The ideas behind quantum mechanics often appear quite strange. In many ways, our everyday experience with the macroscopic physical world does not prepare us for the microscopic world of quantum mechanics. The purpose of this chapter is to introduce you to this exciting world.

Pictured above is a quantum-computer processor. This device is the "brain" of a quantum computer that operates at near-absolute zero temperatures. Unlike a digital computer, which encodes information in binary digits (definite states of either zero or one), a quantum computer encodes information in quantum bits or qubits (mixed states of zero *and* one). Quantum computers are discussed in the first section of this chapter.

7.1 | Wave Functions

Learning Objectives

By the end of this section, you will be able to:

- Describe the statistical interpretation of the wave function
- Use the wave function to determine probabilities
- Calculate expectation values of position, momentum, and kinetic energy

In the preceding chapter, we saw that particles act in some cases like particles and in other cases like waves. But what does it mean for a particle to "act like a wave"? What precisely is "waving"? What rules govern how this wave changes and propagates? How is the wave function used to make predictions? For example, if the amplitude of an electron wave is given by a function of position and time, $\Psi(x,\ t)$, defined for all x, *where* exactly is the electron? The purpose of this chapter is to answer these questions.

Using the Wave Function

A clue to the physical meaning of the wave function $\Psi(x,\ t)$ is provided by the two-slit interference of monochromatic light (**Figure 7.2**). (See also **Electromagnetic Waves (http://cnx.org/content/m58495/latest/)** and **Interference**.) The **wave function** of a light wave is given by $E(x,t)$, and its energy density is given by $|E|^2$, where E is the electric field strength. The energy of an individual photon depends only on the frequency of light, $\varepsilon_{\text{photon}} = hf$, so $|E|^2$ is proportional to the number of photons. When light waves from S_1 interfere with light waves from S_2 at the viewing screen (a distance D away), an interference pattern is produced (part (a) of the figure). Bright fringes correspond to points of constructive interference of the light waves, and dark fringes correspond to points of destructive interference of the light waves (part (b)).

Suppose the screen is initially unexposed to light. If the screen is exposed to very weak light, the interference pattern appears gradually (**Figure 7.2**(c), left to right). Individual photon hits on the screen appear as dots. The dot density is expected to be large at locations where the interference pattern will be, ultimately, the most intense. In other words, the probability (per unit area) that a single photon will strike a particular spot on the screen is proportional to the square of the total electric field, $|E|^2$ at that point. Under the right conditions, the same interference pattern develops for matter particles, such as electrons.

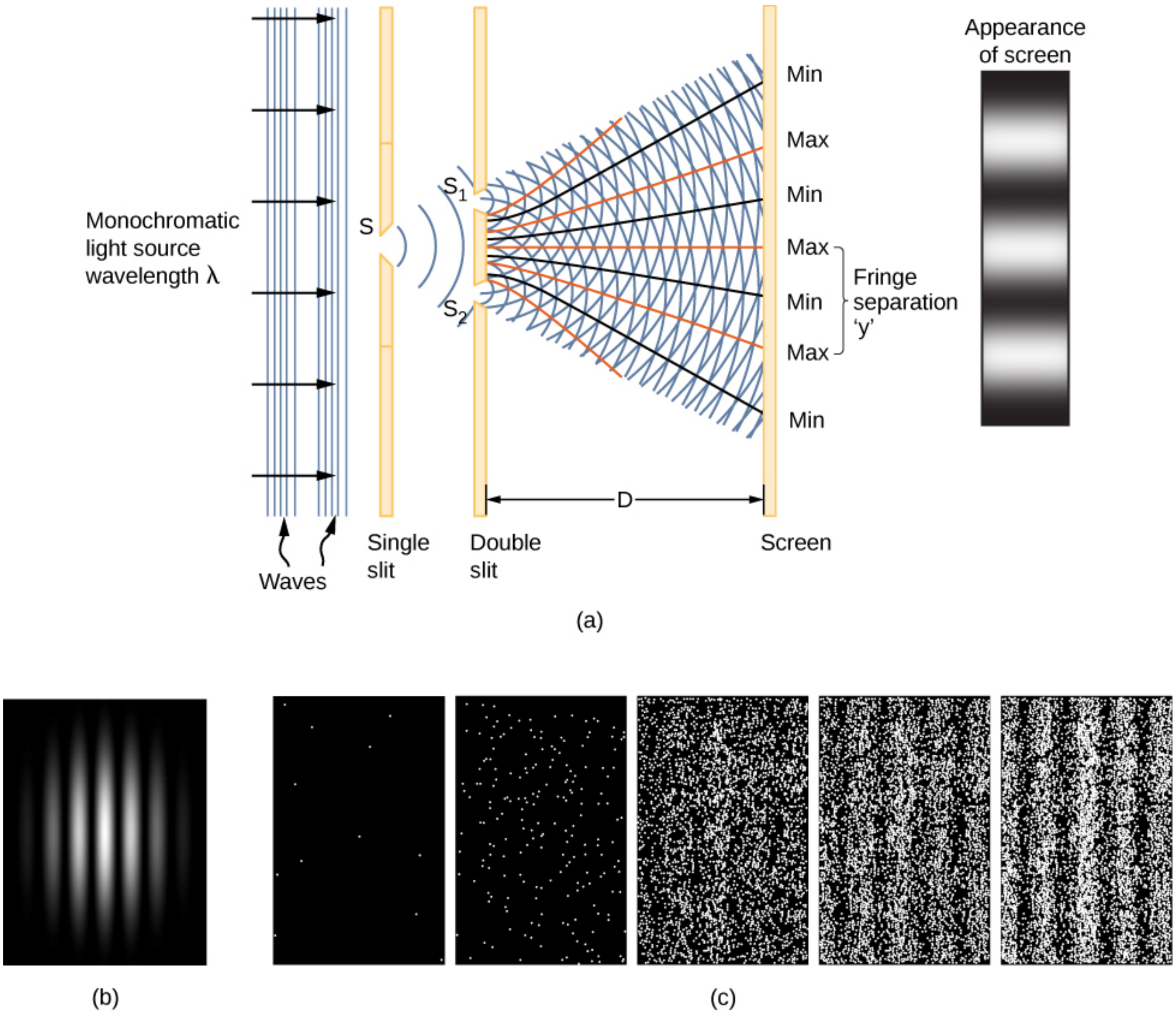


Figure 7.2 Two-slit interference of monochromatic light. (a) Schematic of two-slit interference; (b) light interference pattern; (c) interference pattern built up gradually under low-intensity light (left to right).

Visit this **interactive simulation (https://openstaxcollege.org/l/21intquawavint)** to learn more about quantum wave interference.

The square of the matter wave $|\Psi|^2$ in one dimension has a similar interpretation as the square of the electric field $|E|^2$. It gives the probability that a particle will be found at a particular position and time per unit length, also called the **probability density**. The probability (P) a particle is found in a narrow interval (x, $x + dx$) at time t is therefore

$$P(x,\ x + dx) = |\Psi(x,\ t)|^2 dx. \tag{7.1}$$

(Later, we define the magnitude squared for the general case of a function with "imaginary parts.") This probabilistic interpretation of the wave function is called the **Born interpretation**. Examples of wave functions and their squares for a particular time t are given in **Figure 7.3**.

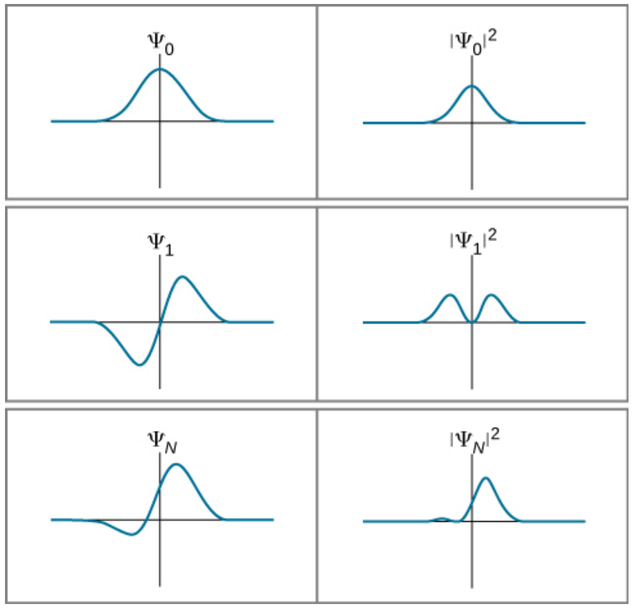


Figure 7.3 Several examples of wave functions and the corresponding square of their wave functions.

If the wave function varies slowly over the interval Δx, the probability a particle is found in the interval is approximately

$$P(x,\ x+\Delta x) \approx |\Psi(x,\ t)|^2 \Delta x. \tag{7.2}$$

Notice that squaring the wave function ensures that the probability is positive. (This is analogous to squaring the electric field strength—which may be positive or negative—to obtain a positive value of intensity.) However, if the wave function does not vary slowly, we must integrate:

$$P(x,\ x+\Delta x) = \int_x^{x+\Delta x} |\Psi(x,\ t)|^2 dx. \tag{7.3}$$

This probability is just the area under the function $|\Psi(x,\ t)|^2$ between *x* and $x+\Delta x$. The probability of finding the particle "somewhere" (the **normalization condition**) is

$$P(-\infty,\ +\infty) = \int_{-\infty}^{\infty} |\Psi(x,\ t)|^2 dx = 1. \tag{7.4}$$

For a particle in two dimensions, the integration is over an area and requires a double integral; for a particle in three dimensions, the integration is over a volume and requires a triple integral. For now, we stick to the simple one-dimensional case.

Example 7.1

Where Is the Ball? (Part I)

A ball is constrained to move along a line inside a tube of length *L*. The ball is equally likely to be found anywhere in the tube at some time *t*. What is the probability of finding the ball in the left half of the tube at that time? (The answer is 50%, of course, but how do we get this answer by using the probabilistic interpretation of the quantum mechanical wave function?)

Strategy

The first step is to write down the wave function. The ball is equally like to be found anywhere in the box, so one way to describe the ball with a *constant* wave function (**Figure 7.4**). The normalization condition can be used to find the value of the function and a simple integration over half of the box yields the final answer.

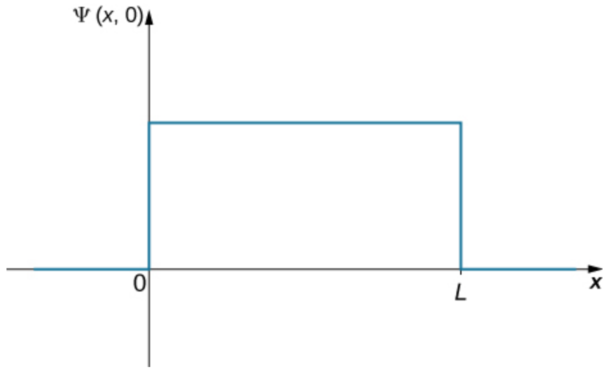


Figure 7.4 Wave function for a ball in a tube of length L.

Solution

The wave function of the ball can be written as $\Psi(x, t) = C(0 < x < L)$, where C is a constant, and $\Psi(x, t) = 0$ otherwise. We can determine the constant C by applying the normalization condition (we set $t = 0$ to simplify the notation):

$$P(x = -\infty, +\infty) = \int_{-\infty}^{\infty} |C|^2 dx = 1.$$

This integral can be broken into three parts: (1) negative infinity to zero, (2) zero to L, and (3) L to infinity. The particle is constrained to be in the tube, so $C = 0$ outside the tube and the first and last integrations are zero. The above equation can therefore be written

$$P(x = 0, L) = \int_0^L |C|^2 dx = 1.$$

The value C does not depend on x and can be taken out of the integral, so we obtain

$$|C|^2 \int_0^{\mathrm{L}} dx = 1.$$

Integration gives

$$C = \sqrt{\frac{1}{L}}.$$

To determine the probability of finding the ball in the first half of the box $(0 < x < L)$, we have

$$P(x = 0, L/2) = \int_0^{L/2} \left| \sqrt{\frac{1}{L}} \right|^2 dx = \left(\frac{1}{L}\right)\frac{L}{2} = 0.50.$$

Significance

The probability of finding the ball in the first half of the tube is 50%, as expected. Two observations are noteworthy. First, this result corresponds to the area under the constant function from $x = 0$ to $L/2$ (the area of a square left of $L/2$). Second, this calculation requires an integration of the *square* of the wave function. A common mistake in performing such calculations is to forget to square the wave function before integration.

Example 7.2

Where Is the Ball? (Part II)

A ball is again constrained to move along a line inside a tube of length L. This time, the ball is found preferentially in the middle of the tube. One way to represent its wave function is with a simple cosine function (**Figure 7.5**). What is the probability of finding the ball in the last one-quarter of the tube?

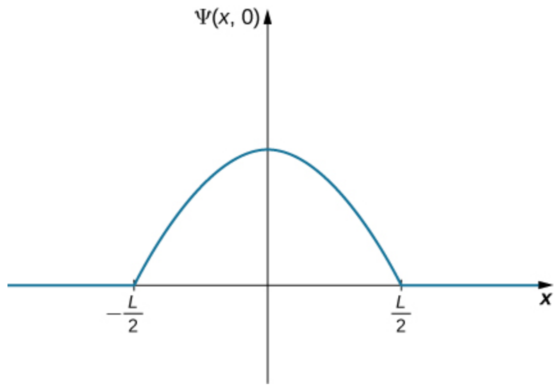


Figure 7.5 Wave function for a ball in a tube of length L, where the ball is preferentially in the middle of the tube.

Strategy

We use the same strategy as before. In this case, the wave function has two unknown constants: One is associated with the wavelength of the wave and the other is the amplitude of the wave. We determine the amplitude by using the boundary conditions of the problem, and we evaluate the wavelength by using the normalization condition. Integration of the square of the wave function over the last quarter of the tube yields the final answer. The calculation is simplified by centering our coordinate system on the peak of the wave function.

Solution

The wave function of the ball can be written

$$\Psi(x, 0) = A\cos(kx)(-L/2 < x < L/2),$$

where A is the amplitude of the wave function and $k = 2\pi/\lambda$ is its wave number. Beyond this interval, the amplitude of the wave function is zero because the ball is confined to the tube. Requiring the wave function to terminate at the right end of the tube gives

$$\Psi\left(x = \frac{L}{2}, 0\right) = 0.$$

Evaluating the wave function at $x = L/2$ gives

$$A\cos(kL/2) = 0.$$

This equation is satisfied if the argument of the cosine is an integral multiple of $\pi/2,\ 3\pi/2,\ 5\pi/2,$ and so on. In this case, we have

$$\frac{kL}{2} = \frac{\pi}{2},$$

or

$$k = \frac{\pi}{L}.$$

Applying the normalization condition gives $A = \sqrt{2/L}$, so the wave function of the ball is

$$\Psi(x, 0) = \sqrt{\frac{2}{L}}\cos(\pi x/L), \quad -L/2 < x < L/2.$$

To determine the probability of finding the ball in the last quarter of the tube, we square the function and integrate:

$$P(x = L/4,\ L/2) = \int_{L/4}^{L/2} \left| \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right) \right|^2 dx = 0.091.$$

Significance

The probability of finding the ball in the last quarter of the tube is 9.1%. The ball has a definite wavelength $(\lambda = 2L)$. If the tube is of macroscopic length $(L = 1\text{ m})$, the momentum of the ball is

$$p = \frac{h}{\lambda} = \frac{h}{2L} \sim 10^{-36}\text{ m/s}.$$

This momentum is much too small to be measured by any human instrument.

An Interpretation of the Wave Function

We are now in position to begin to answer the questions posed at the beginning of this section. First, for a traveling particle described by $\Psi(x,\ t) = A\ \sin(kx - \omega t)$, what is "waving?" Based on the above discussion, the answer is a mathematical function that can, among other things, be used to determine where the particle is likely to be when a position measurement is performed. Second, how is the wave function used to make predictions? If it is necessary to find the probability that a particle will be found in a certain interval, square the wave function and integrate over the interval of interest. Soon, you will learn soon that the wave function can be used to make many other kinds of predictions, as well.

Third, if a matter wave is given by the wave function $\Psi(x,\ t)$, *where* exactly is the particle? Two answers exist: (1) when the observer *is not* looking (or the particle is not being otherwise detected), the particle is everywhere $(x = -\infty,\ +\infty)$; and (2) when the observer *is* looking (the particle is being detected), the particle "jumps into" a particular position state $(x,\ x + dx)$ with a probability given by $P(x,\ x + dx) = |\Psi(x,\ t)|^2 dx$ —a process called **state reduction** or **wave function collapse**. This answer is called the **Copenhagen interpretation** of the wave function, or of quantum mechanics.

To illustrate this interpretation, consider the simple case of a particle that can occupy a small container either at x_1 or x_2 (**Figure 7.6**). In classical physics, we assume the particle is located either at x_1 or x_2 when the observer is not looking. However, in quantum mechanics, the particle may exist in a state of indefinite position—that is, it may be located at x_1 *and* x_2 when the observer is not looking. The assumption that a particle can only have one value of position (when the observer is not looking) is abandoned. Similar comments can be made of other measurable quantities, such as momentum and energy.

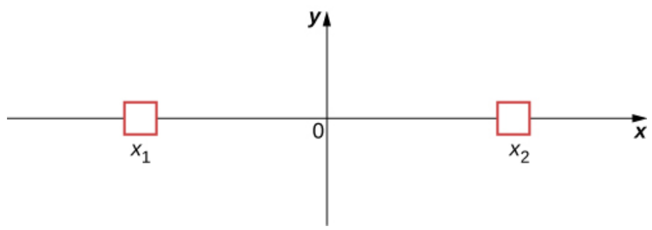


Figure 7.6 A two-state system of position of a particle.

The bizarre consequences of the Copenhagen interpretation of quantum mechanics are illustrated by a creative thought experiment first articulated by Erwin Schrödinger (*National Geographic*, 2013) (**Figure 7.7**):

"A cat is placed in a steel box along with a Geiger counter, a vial of poison, a hammer, and a radioactive substance. When the radioactive substance decays, the Geiger detects it and triggers the hammer to release the poison, which subsequently kills the cat. The radioactive decay is a random [probabilistic] process, and there is no way to predict when it will happen. Physicists say the atom exists in a state known as a superposition—both decayed and not decayed at the same time. Until the box is opened, an observer doesn't know whether the cat is alive or dead—because the cat's fate is intrinsically tied to whether or not the atom has decayed and the cat would [according to the Copenhagen interpretation] be "living and dead ... in equal parts" until it is observed."

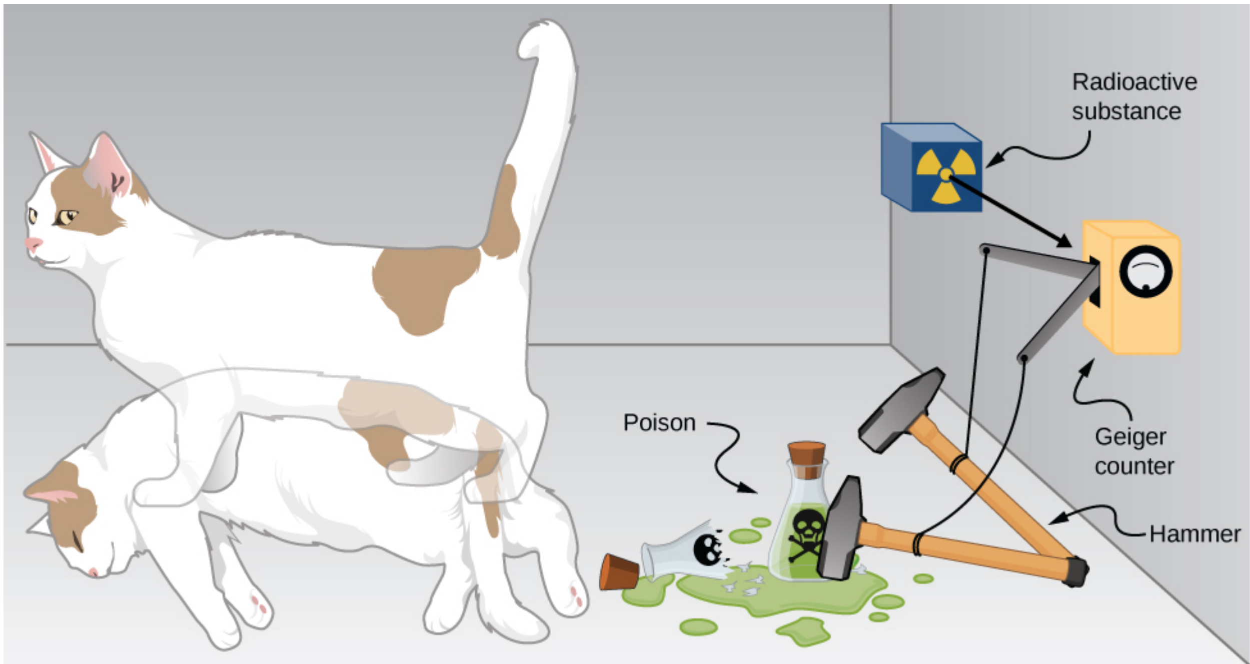


Figure 7.7 Schrödinger's cat.

Schrödinger took the absurd implications of this thought experiment (a cat simultaneously dead and alive) as an argument against the Copenhagen interpretation. However, this interpretation remains the most commonly taught view of quantum mechanics.

Two-state systems (left and right, atom decays and does not decay, and so on) are often used to illustrate the principles of quantum mechanics. These systems find many applications in nature, including electron spin and mixed states of particles, atoms, and even molecules. Two-state systems are also finding application in the quantum computer, as mentioned in the introduction of this chapter. Unlike a digital computer, which encodes information in binary digits (zeroes and ones), a quantum computer stores and manipulates data in the form of quantum bits, or qubits. In general, a qubit is not in a state of zero or one, but rather in a mixed state of zero *and* one. If a large number of qubits are placed in the same quantum state, the measurement of an individual qubit would produce a zero with a probability p, and a one with a probability $q = 1 - p$.

Many scientists believe that quantum computers are the future of the computer industry.

Complex Conjugates

Later in this section, you will see how to use the wave function to describe particles that are "free" or bound by forces to other particles. The specific form of the wave function depends on the details of the physical system. A peculiarity of quantum theory is that these functions are usually **complex functions**. A complex function is one that contains one or more imaginary numbers $(i = \sqrt{-1})$. Experimental measurements produce real (nonimaginary) numbers only, so the above procedure to use the wave function must be slightly modified. In general, the probability that a particle is found in the narrow interval $(x, x + dx)$ at time t is given by

$$P(x,\ x + dx) = |\Psi(x,\ t)|^2 dx = \Psi^*\ (x,\ t)\Psi(x,\ t)dx, \tag{7.5}$$

where $\Psi^*\ (x,\ t)$ is the complex conjugate of the wave function. The complex conjugate of a function is obtaining by replacing every occurrence of $i = \sqrt{-1}$ in that function with $-i$. This procedure eliminates complex numbers in all predictions because the product $\Psi^*\ (x,\ t)\Psi(x,\ t)$ is always a real number.

7.1 Check Your Understanding If $a = 3 + 4i$, what is the product $a^*\ a$?

Consider the motion of a free particle that moves along the *x*-direction. As the name suggests, a free particle experiences no forces and so moves with a constant velocity. As we will see in a later section of this chapter, a formal quantum mechanical treatment of a free particle indicates that its wave function has real *and* complex parts. In particular, the wave function is given by

$$\Psi(x,\ t) = A\cos(kx - \omega t) + iA\sin(kx - \omega t),$$

where A is the amplitude, k is the wave number, and ω is the angular frequency. Using Euler's formula, $e^{i\phi} = \cos(\phi) + i\sin(\phi)$, this equation can be written in the form

$$\Psi(x,\ t) = Ae^{i(kx - \omega t)} = Ae^{i\phi},$$

where ϕ is the phase angle. If the wave function varies slowly over the interval Δx, the probability of finding the particle in that interval is

$$P(x,\ x + \Delta x) \approx \Psi^*(x,\ t)\Psi(x,\ t)\Delta x = \left(Ae^{i\phi}\right)\left(A^* e^{-i\phi}\right)\Delta x = (A^* A)\Delta x.$$

If A has real and complex parts $(a + ib$, where a and b are real constants), then

$$A^* A = (a + ib)(a - ib) = a^2 + b^2.$$

Notice that the complex numbers have vanished. Thus,

$$P(x,\ x + \Delta x) \approx |A|^2 \Delta x$$

is a real quantity. The interpretation of $\Psi^*(x,\ t)\Psi(x,\ t)$ as a probability density ensures that the predictions of quantum mechanics can be checked in the "real world."

7.2 Check Your Understanding Suppose that a particle with energy E is moving along the *x*-axis and is confined in the region between 0 and L. One possible wave function is

$$\psi(x,\ t) = \begin{cases} Ae^{-iEt/\hbar}\sin\dfrac{\pi x}{L}, & \text{when } 0 \le x \le L \\ 0, & \text{otherwise} \end{cases}.$$

Determine the normalization constant.

Expectation Values

In classical mechanics, the solution to an equation of motion is a function of a measurable quantity, such as $x(t)$, where x is the position and t is the time. Note that the particle has one value of position for any time t. In quantum mechanics, however, the solution to an equation of motion is a wave function, $\Psi(x,\ t)$. The particle has many values of position for any time t, and only the probability density of finding the particle, $|\Psi(x,\ t)|^2$, can be known. The average value of position for a large number of particles with the same wave function is expected to be

$$\langle x \rangle = \int_{-\infty}^{\infty} xP(x,\ t)dx = \int_{-\infty}^{\infty} x\Psi^*(x,\ t)\Psi(x,\ t)dx. \tag{7.6}$$

This is called the **expectation value** of the position. It is usually written

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^*(x,\ t)x\Psi(x,\ t)dx, \tag{7.7}$$

where the x is sandwiched between the wave functions. The reason for this will become apparent soon. Formally, x is called the **position operator**.

At this point, it is important to stress that a wave function can be written in terms of other quantities as well, such as velocity (v), momentum (p), and kinetic energy (K). The expectation value of momentum, for example, can be written

$$\langle p \rangle = \int_{-\infty}^{\infty} \Psi^* (p, t) p \Psi(p, t) dp, \tag{7.8}$$

Where dp is used instead of dx to indicate an infinitesimal interval in momentum. In some cases, we know the wave function in position, $\Psi(x, t)$, but seek the expectation of momentum. The procedure for doing this is

$$\langle p \rangle = \int_{-\infty}^{\infty} \Psi^* (x, t)\left(-i\hbar \frac{d}{dx}\right)\Psi(x, t)dx, \tag{7.9}$$

where the quantity in parentheses, sandwiched between the wave functions, is called the **momentum operator** in the x-direction. [The momentum operator in Equation 7.9 is said to be the position-space representation of the momentum operator.] The momentum operator must act (operate) on the wave function to the right, and then the result must be multiplied by the complex conjugate of the wave function on the left, before integration. The momentum operator in the x-direction is sometimes denoted

$$(p_x)_{\text{op}} = -i\hbar \frac{d}{dx}, \tag{7.10}$$

Momentum operators for the y- and z-directions are defined similarly. This operator and many others are derived in a more advanced course in modern physics. In some cases, this derivation is relatively simple. For example, the kinetic energy operator is just

$$(K)_{\text{op}} = \frac{1}{2}m(v_x)^2_{\text{op}} = \frac{(p_x)^2_{\text{op}}}{2m} = \frac{\left(-i\hbar \frac{d}{dx}\right)^2}{2m} = \frac{-\hbar^2}{2m}\left(\frac{d}{dx}\right)\left(\frac{d}{dx}\right). \tag{7.11}$$

Thus, if we seek an expectation value of kinetic energy of a particle in one dimension, two successive ordinary derivatives of the wave function are required before integration.

Expectation-value calculations are often simplified by exploiting the symmetry of wave functions. Symmetric wave functions can be even or odd. An **even function** is a function that satisfies

$$\psi(x) = \psi(-x). \tag{7.12}$$

In contrast, an **odd function** is a function that satisfies

$$\psi(x) = -\psi(-x). \tag{7.13}$$

An example of even and odd functions is shown in Figure 7.8. An even function is symmetric about the y-axis. This function is produced by reflecting $\psi(x)$ for $x > 0$ about the vertical y-axis. By comparison, an odd function is generated by reflecting the function about the y-axis and then about the x-axis. (An odd function is also referred to as an **anti-symmetric function**.)

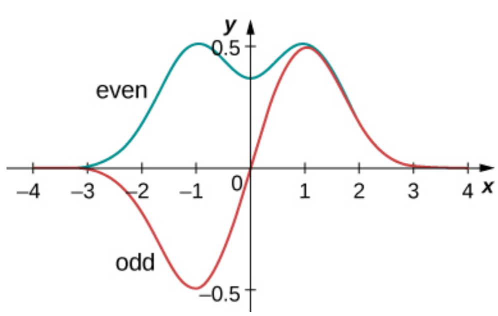


Figure 7.8 Examples of even and odd wave functions.

In general, an even function times an even function produces an even function. A simple example of an even function is the product $x^2 e^{-x^2}$ (even times even is even). Similarly, an odd function times an odd function produces an even function, such as $x \sin x$ (odd times odd is even). However, an odd function times an even function produces an odd function, such as xe^{-x^2} (odd times even is odd). The integral over all space of an odd function is zero, because the total area of the function above the x-axis cancels the (negative) area below it. As the next example shows, this property of odd functions is very useful.

Example 7.3

Expectation Value (Part I)

The normalized wave function of a particle is

$$\psi(x) = e^{-|x|/x_0}/\sqrt{x_0}.$$

Find the expectation value of position.

Strategy

Substitute the wave function into **Equation 7.7** and evaluate. The position operator introduces a multiplicative factor only, so the position operator need not be "sandwiched."

Solution

First multiply, then integrate:

$$\langle x \rangle = \int_{-\infty}^{+\infty} dx x|\psi(x)|^2 = \int_{-\infty}^{+\infty} dx x\left|\frac{e^{-|x|/x_0}}{\sqrt{x_0}}\right|^2 = \frac{1}{x_0}\int_{-\infty}^{+\infty} dx x e^{-2|x|/x_0} = 0.$$

Significance

The function in the integrand $(xe^{-2|x|/x_0})$ is odd since it is the product of an odd function (x) and an even function $(e^{-2|x|/x_0})$. The integral vanishes because the total area of the function about the x-axis cancels the (negative) area below it. The result $(\langle x \rangle = 0)$ is not surprising since the probability density function is symmetric about $x = 0$.

Example 7.4

Expectation Value (Part II)

The time-dependent wave function of a particle confined to a region between 0 and L is

$$\psi(x, t) = Ae^{-i\omega t}\sin(\pi x/L)$$

where ω is angular frequency and E is the energy of the particle. (*Note:* The function varies as a sine because of the limits (0 to L). When $x = 0,$ the sine factor is zero and the wave function is zero, consistent with the boundary conditions.) Calculate the expectation values of position, momentum, and kinetic energy.

Strategy

We must first normalize the wave function to find A. Then we use the operators to calculate the expectation values.

Solution

Computation of the normalization constant:

$$1 = \int_0^L dx \psi^*(x)\psi(x) = \int_0^L dx\left(Ae^{+i\omega t}\sin\frac{\pi x}{L}\right)\left(Ae^{-i\omega t}\sin\frac{\pi x}{L}\right) = A^2\int_0^L dx \sin^2\frac{\pi x}{L} = A^2\frac{L}{2} \Rightarrow A = \sqrt{\frac{2}{L}}.$$

The expectation value of position is

$$\langle x \rangle = \int_0^L dx\psi^*(x)x\psi(x) = \int_0^L dx\left(Ae^{+i\omega t}\sin\frac{\pi x}{L}\right)x\left(Ae^{-i\omega t}\sin\frac{\pi x}{L}\right) = A^2\int_0^L dx x\sin^2\frac{\pi x}{L} = A^2\frac{L^2}{4} = \frac{L}{2}.$$

The expectation value of momentum in the x-direction also requires an integral. To set this integral up, the associated operator must— by rule—act to the right on the wave function $\psi(x)$:

$$-i\hbar\frac{d}{dx}\psi(x) = -i\hbar\frac{d}{dx}Ae^{-i\omega t}\sin\frac{\pi x}{L} = -i\frac{Ah}{2L}e^{-i\omega t}\cos\frac{\pi x}{L}.$$

Therefore, the expectation value of momentum is

$$\langle p \rangle = \int_0^L dx\left(Ae^{+i\omega t}\sin\frac{\pi x}{L}\right)\left(-i\frac{Ah}{2L}e^{-i\omega t}\cos\frac{\pi x}{L}\right) = -i\frac{A^2h}{4L}\int_0^L dx\sin\frac{2\pi x}{L} = 0.$$

The function in the integral is a sine function with a wavelength equal to the width of the well, L—an odd function about $x = L/2$. As a result, the integral vanishes.

The expectation value of kinetic energy in the x-direction requires the associated operator to act on the wave function:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}Ae^{-i\omega t}\sin\frac{\pi x}{L} = -\frac{\hbar^2}{2m}Ae^{-i\omega t}\frac{d^2}{dx^2}\sin\frac{\pi x}{L} = \frac{Ah^2}{2mL^2}e^{-i\omega t}\sin\frac{\pi x}{L}.$$

Thus, the expectation value of the kinetic energy is

$$\begin{aligned}\langle K \rangle &= \int_0^L dx\left(Ae^{+i\omega t}\sin\frac{\pi x}{L}\right)\left(\frac{Ah^2}{2mL^2}e^{-i\omega t}\sin\frac{\pi x}{L}\right)\\ &= \frac{A^2h^2}{2mL^2}\int_0^L dx\sin^2\frac{\pi x}{L} = \frac{A^2h^2}{2mL^2}\frac{L}{2} = \frac{h^2}{2mL^2}.\end{aligned}$$

Significance

The average position of a large number of particles in this state is $L/2$. The average momentum of these particles is zero because a given particle is equally likely to be moving right or left. However, the particle is not at rest because its average kinetic energy is not zero. Finally, the probability density is

$$|\psi|^2 = (2/L)\sin^2(\pi x/L).$$

This probability density is largest at location $L/2$ and is zero at $x = 0$ and at $x = L$. Note that these conclusions do not depend explicitly on time.

7.3 Check Your Understanding For the particle in the above example, find the probability of locating it between positions 0 and $L/4$

Quantum mechanics makes many surprising predictions. However, in 1920, Niels Bohr (founder of the Niels Bohr Institute in Copenhagen, from which we get the term “Copenhagen interpretation”) asserted that the predictions of quantum mechanics and classical mechanics must agree for all macroscopic systems, such as orbiting planets, bouncing balls, rocking chairs, and springs. This **correspondence principle** is now generally accepted. It suggests the rules of classical mechanics are an approximation of the rules of quantum mechanics for systems with very large energies. Quantum mechanics describes both the microscopic and macroscopic world, but classical mechanics describes only the latter.

7.2 | The Heisenberg Uncertainty Principle

Learning Objectives

By the end of this section, you will be able to:

- Describe the physical meaning of the position-momentum uncertainty relation
- Explain the origins of the uncertainty principle in quantum theory
- Describe the physical meaning of the energy-time uncertainty relation

Heisenberg's uncertainty principle is a key principle in quantum mechanics. Very roughly, it states that if we know *everything* about where a particle is located (the uncertainty of position is small), we know *nothing* about its momentum (the uncertainty of momentum is large), and vice versa. Versions of the uncertainty principle also exist for other quantities as well, such as energy and time. We discuss the momentum-position and energy-time uncertainty principles separately.

Momentum and Position

To illustrate the momentum-position uncertainty principle, consider a free particle that moves along the *x*-direction. The particle moves with a constant velocity u and momentum $p = mu$. According to de Broglie's relations, $p = \hbar k$ and $E = \hbar \omega$. As discussed in the previous section, the wave function for this particle is given by

$$\psi_k(x, t) = A[\cos(\omega t - kx) - i\sin(\omega t - kx)] = Ae^{-i(\omega t - kx)} = Ae^{-i\omega t}e^{ikx} \quad (7.14)$$

and the probability density $\left|\psi_k(x, t)\right|^2 = A^2$ is *uniform* and independent of time. The particle is equally likely to be found anywhere along the *x*-axis but has definite values of wavelength and wave number, and therefore momentum. The uncertainty of position is infinite (we are completely uncertain about position) and the uncertainty of the momentum is zero (we are completely certain about momentum). This account of a free particle is consistent with Heisenberg's uncertainty principle.

Similar statements can be made of localized particles. In quantum theory, a localized particle is modeled by a linear superposition of free-particle (or plane-wave) states called a **wave packet**. An example of a wave packet is shown in **Figure 7.9**. A wave packet contains many wavelengths and therefore by de Broglie's relations many momenta—possible in quantum mechanics! This particle also has many values of position, although the particle is confined mostly to the interval Δx. The particle can be better localized $(\Delta x$ can be decreased) if more plane-wave states of different wavelengths or momenta are added together in the right way $(\Delta p$ is increased). According to Heisenberg, these uncertainties obey the following relation.

The Heisenberg Uncertainty Principle

The product of the uncertainty in position of a particle and the uncertainty in its momentum can never be less than one-half of the reduced Planck constant:

$$\Delta x \Delta p \geq \hbar/2. \quad (7.15)$$

This relation expresses Heisenberg's uncertainty principle. It places limits on what we can know about a particle from simultaneous measurements of position and momentum. If Δx is large, Δp is small, and vice versa. **Equation 7.15** can be derived in a more advanced course in modern physics. Reflecting on this relation in his work *The Physical Principles of the Quantum Theory*, Heisenberg wrote "Any use of the words 'position' and 'velocity' with accuracy exceeding that given by [the relation] is just as meaningless as the use of words whose sense is not defined."

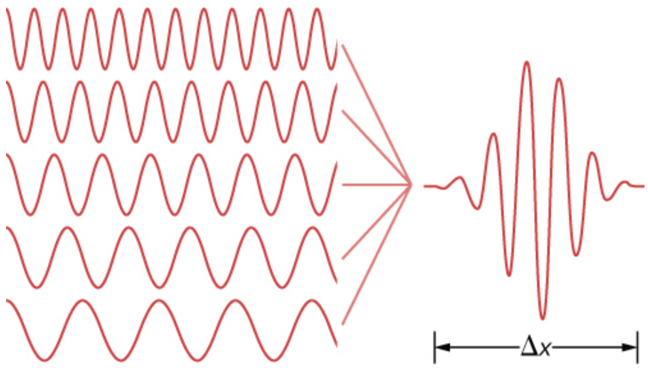


Figure 7.9 Adding together several plane waves of different wavelengths can produce a wave that is relatively localized.

Note that the uncertainty principle has nothing to do with the precision of an experimental apparatus. Even for perfect measuring devices, these uncertainties would remain because they originate in the wave-like nature of matter. The precise value of the product $\Delta x \Delta p$ depends on the specific form of the wave function. Interestingly, the Gaussian function (or bell-curve distribution) gives the minimum value of the uncertainty product: $\Delta x \Delta p = \hbar/2$.

Example 7.5

The Uncertainty Principle Large and Small

Determine the minimum uncertainties in the positions of the following objects if their speeds are known with a precision of $1.0 \times 10^{-3}\,\text{m/s}$: (a) an electron and (b) a bowling ball of mass 6.0 kg.

Strategy

Given the uncertainty in speed $\Delta u = 1.0 \times 10^{-3}\,\text{m/s}$, we have to first determine the uncertainty in momentum $\Delta p = m\Delta u$ and then invert **Equation 7.15** to find the uncertainty in position $\Delta x = \hbar/(2\Delta p)$.

Solution

a. For the electron:

$$\begin{aligned} \Delta p &= m\Delta u = (9.1 \times 10^{-31}\,\text{kg})(1.0 \times 10^{-3}\,\text{m/s}) = 9.1 \times 10^{-34}\,\text{kg} \cdot \text{m/s}, \\ \Delta x &= \frac{\hbar}{2\Delta p} = 5.8\,\text{cm}. \end{aligned}$$

b. For the bowling ball:

$$\begin{aligned} \Delta p &= m\Delta u = (6.0\,\text{kg})(1.0 \times 10^{-3}\,\text{m/s}) = 6.0 \times 10^{-3}\,\text{kg} \cdot \text{m/s}, \\ \Delta x &= \frac{\hbar}{2\Delta p} = 8.8 \times 10^{-33}\,\text{m}. \end{aligned}$$

Significance

Unlike the position uncertainty for the electron, the position uncertainty for the bowling ball is immeasurably small. Planck's constant is very small, so the limitations imposed by the uncertainty principle are not noticeable in macroscopic systems such as a bowling ball.

Example 7.6

Uncertainty and the Hydrogen Atom

Estimate the ground-state energy of a hydrogen atom using Heisenberg's uncertainty principle. (*Hint*: According to early experiments, the size of a hydrogen atom is approximately 0.1 nm.)

Strategy

An electron bound to a hydrogen atom can be modeled by a particle bound to a one-dimensional box of length $L = 0.1\text{ nm}$. The ground-state wave function of this system is a half wave, like that given in **Example 7.1**. This is the largest wavelength that can "fit" in the box, so the wave function corresponds to the lowest energy state. Note that this function is very similar in shape to a Gaussian (bell curve) function. We can take the average energy of a particle described by this function (E) as a good estimate of the ground state energy (E_0). This average energy of a particle is related to its average of the momentum squared, which is related to its momentum uncertainty.

Solution

To solve this problem, we must be specific about what is meant by "uncertainty of position" and "uncertainty of momentum." We identify the uncertainty of position (Δx) with the standard deviation of position (σ_x), and the uncertainty of momentum (Δp) with the standard deviation of momentum (σ_p). For the Gaussian function, the uncertainty product is

$$\sigma_x \sigma_p = \frac{\hbar}{2},$$

where

$$\sigma_x^2 = \overline{x^2} - \overline{x}^2 \text{ and } \sigma_p^2 = \overline{p^2} - \overline{p}^2.$$

The particle is equally likely to be moving left as moving right, so $\overline{p} = 0$. Also, the uncertainty of position is comparable to the size of the box, so $\sigma_x = L$. The estimated ground state energy is therefore

$$E_0 = E_{\text{Gaussian}} = \frac{\overline{p^2}}{m} = \frac{\sigma_p^2}{2m} = \frac{1}{2m}\left(\frac{\hbar}{2\sigma_x}\right)^2 = \frac{1}{2m}\left(\frac{\hbar}{2L}\right)^2 = \frac{\hbar^2}{8mL^2}.$$

Multiplying numerator and denominator by c^2 gives

$$E_0 = \frac{(\hbar c)^2}{8(mc^2)L^2} = \frac{(197.3\text{ eV}\cdot\text{nm})^2}{8\left(0.511\cdot 10^6\text{ eV}\right)(0.1\text{ nm})^2} = 0.952\text{ eV} \approx 1\text{ eV}.$$

Significance

Based on early estimates of the size of a hydrogen atom and the uncertainty principle, the ground-state energy of a hydrogen atom is in the eV range. The ionization energy of an electron in the ground-state energy is approximately 10 eV, so this prediction is roughly confirmed. (*Note:* The product $\hbar c$ is often a useful value in performing calculations in quantum mechanics.)

Energy and Time

Another kind of uncertainty principle concerns uncertainties in simultaneous measurements of the energy of a quantum state and its lifetime,

$$\Delta E \Delta t \geq \frac{\hbar}{2}, \quad (7.16)$$

where ΔE is the uncertainty in the energy measurement and Δt is the uncertainty in the lifetime measurement. The **energy-time uncertainty principle** does not result from a relation of the type expressed by **Equation 7.15** for technical reasons beyond this discussion. Nevertheless, the general meaning of the energy-time principle is that a quantum state that exists for only a short time cannot have a definite energy. The reason is that the frequency of a state is inversely proportional to time and the frequency connects with the energy of the state, so to measure the energy with good precision, the state must be observed for many cycles.

To illustrate, consider the excited states of an atom. The finite lifetimes of these states can be deduced from the shapes of spectral lines observed in atomic emission spectra. Each time an excited state decays, the emitted energy is slightly different and, therefore, the emission line is characterized by a *distribution* of spectral frequencies (or wavelengths) of the emitted photons. As a result, all spectral lines are characterized by spectral widths. The average energy of the emitted photon corresponds to the theoretical energy of the excited state and gives the spectral location of the peak of the emission line. Short-lived states have broad spectral widths and long-lived states have narrow spectral widths.

Example 7.7

Atomic Transitions

An atom typically exists in an excited state for about $\Delta t = 10^{-8}\,\text{s}$. Estimate the uncertainty Δf in the frequency of emitted photons when an atom makes a transition from an excited state with the simultaneous emission of a photon with an average frequency of $f = 7.1 \times 10^{14}\ \text{Hz}$. Is the emitted radiation monochromatic?

Strategy

We invert **Equation 7.16** to obtain the energy uncertainty $\Delta E \approx \hbar/2\Delta t$ and combine it with the photon energy $E = hf$ to obtain Δf. To estimate whether or not the emission is monochromatic, we evaluate $\Delta f/f$.

Solution

The spread in photon energies is $\Delta E = h\Delta f$. Therefore,

$$\Delta E \approx \frac{\hbar}{2\Delta t} \Rightarrow h\Delta f \approx \frac{\hbar}{2\Delta t} \Rightarrow \Delta f \approx \frac{1}{4\pi\Delta t} = \frac{1}{4\pi(10^{-8}\,\text{s})} = 8.0 \times 10^{6}\ \text{Hz},$$

$$\frac{\Delta f}{f} = \frac{8.0 \times 10^{6}\ \text{Hz}}{7.1 \times 10^{14}\ \text{Hz}} = 1.1 \times 10^{-8}.$$

Significance

Because the emitted photons have their frequencies within 1.1×10^{-6} percent of the average frequency, the emitted radiation can be considered monochromatic.

7.4 Check Your Understanding A sodium atom makes a transition from the first excited state to the ground state, emitting a 589.0-nm photon with energy 2.105 eV. If the lifetime of this excited state is $1.6 \times 10^{-8}\,\text{s}$, what is the uncertainty in energy of this excited state? What is the width of the corresponding spectral line?

7.3 | The Schrödinger Equation

Learning Objectives

By the end of this section, you will be able to:

- Describe the role Schrödinger's equation plays in quantum mechanics
- Explain the difference between time-dependent and -independent Schrödinger's equations
- Interpret the solutions of Schrödinger's equation

In the preceding two sections, we described how to use a quantum mechanical wave function and discussed Heisenberg's uncertainty principle. In this section, we present a complete and formal theory of quantum mechanics that can be used to make predictions. In developing this theory, it is helpful to review the wave theory of light. For a light wave, the electric field $E(x,t)$ obeys the relation

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}, \tag{7.17}$$

where c is the speed of light and the symbol ∂ represents a *partial derivative*. (Recall from **Oscillations (http://cnx.org/content/m58360/latest/)** that a partial derivative is closely related to an ordinary derivative, but involves functions of more than one variable. When taking the partial derivative of a function by a certain variable, all other variables are held constant.) A light wave consists of a very large number of photons, so the quantity $|E(x,\ t)|^2$ can interpreted as a probability density of finding a single photon at a particular point in space (for example, on a viewing screen).

There are many solutions to this equation. One solution of particular importance is

$$E(x,\ t) = A\sin(kx - \omega t), \tag{7.18}$$

where A is the amplitude of the electric field, k is the wave number, and ω is the angular frequency. Combing this equation with **Equation 7.17** gives

$$k^2 = \frac{\omega^2}{c^2}. \tag{7.19}$$

According to de Broglie's equations, we have $p = \hbar k$ and $E = \hbar\omega$. Substituting these equations in **Equation 7.19** gives

$$p = \frac{E}{c}, \tag{7.20}$$

or

$$E = pc. \tag{7.21}$$

Therefore, according to Einstein's general energy-momentum equation (**Equation 5.11**), **Equation 7.17** describes a particle with a zero rest mass. This is consistent with our knowledge of a photon.

This process can be reversed. We can begin with the energy-momentum equation of a particle and then ask what wave equation corresponds to it. The energy-momentum equation of a nonrelativistic particle in one dimension is

$$E = \frac{p^2}{2m} + U(x,\ t), \tag{7.22}$$

where p is the momentum, m is the mass, and U is the potential energy of the particle. The wave equation that goes with it turns out to be a key equation in quantum mechanics, called **Schrödinger's time-dependent equation**.

The Schrödinger Time-Dependent Equation

The equation describing the energy and momentum of a wave function is known as the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,\ t)}{\partial x^2} + U(x,\ t)\Psi(x,\ t) = i\hbar\frac{\partial\Psi(x,\ t)}{\partial t}. \tag{7.23}$$

As described in **Potential Energy and Conservation of Energy (http://cnx.org/content/m58311/latest/)** , the force on the particle described by this equation is given by

$$F = -\frac{\partial U(x,\ t)}{\partial x}. \tag{7.24}$$

This equation plays a role in quantum mechanics similar to Newton's second law in classical mechanics. Once the potential energy of a particle is specified—or, equivalently, once the force on the particle is specified—we can solve this differential equation for the wave function. The solution to Newton's second law equation (also a differential equation) in one dimension is a function $x(t)$ that specifies where an object is at any time t. The solution to Schrödinger's time-dependent equation provides a tool—the wave function—that can be used to determine where the particle is *likely* to be. This equation can be also written in two or three dimensions. Solving Schrödinger's time-dependent equation often requires the aid of a computer.

Consider the special case of a free particle. A free particle experiences no force $(F = 0)$. Based on **Equation 7.24**, this requires only that

$$U(x,\ t) = U_0 = \text{constant}. \tag{7.25}$$

For simplicity, we set $U_0 = 0$. Schrödinger's equation then reduces to

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,\ t)}{\partial x^2} = i\hbar\frac{\partial\Psi(x,\ t)}{\partial t}. \tag{7.26}$$

A valid solution to this equation is

$$\Psi(x,\ t) = Ae^{i(kx - \omega t)}. \tag{7.27}$$

Not surprisingly, this solution contains an imaginary number $(i = \sqrt{-1})$ because the differential equation itself contains an imaginary number. As stressed before, however, quantum-mechanical predictions depend only on $|\Psi(x,\ t)|^2$, which yields completely real values. Notice that the real plane-wave solutions, $\Psi(x,\ t) = A\sin(kx - \omega t)$ and $\Psi(x,\ t) = A\cos(kx - \omega t)$, do not obey Schrödinger's equation. The temptation to think that a wave function can be seen, touched, and felt in nature is eliminated by the appearance of an imaginary number. In Schrödinger's theory of quantum mechanics, the wave function is merely a tool for calculating things.

If the potential energy function (U) does not depend on time, it is possible to show that

$$\Psi(x,\ t) = \psi(x)e^{-i\omega t} \tag{7.28}$$

satisfies Schrödinger's time-dependent equation, where $\psi(x)$ is a *time*-independent function and $e^{-i\omega t}$ is a *space*-independent function. In other words, the wave function is *separable* into two parts: a space-only part and a time-only part. The factor $e^{-i\omega t}$ is sometimes referred to as a **time-modulation factor** since it modifies the space-only function. According to de Broglie, the energy of a matter wave is given by $E = \hbar\omega$, where E is its total energy. Thus, the above equation can also be written as

$$\Psi(x,\ t) = \psi(x)e^{-iEt/\hbar}. \tag{7.29}$$

Any linear combination of such states (mixed state of energy or momentum) is also valid solution to this equation. Such states can, for example, describe a localized particle (see **Figure 7.9**)

7.5 Check Your Understanding A particle with mass m is moving along the x-axis in a potential given by the potential energy function $U(x) = 0.5m\omega^2 x^2$. Compute the product $\Psi(x,\ t)^*\ U(x)\Psi(x,\ t)$. Express your answer in terms of the time-independent wave function, $\psi(x)$.

Combining **Equation 7.23** and **Equation 7.28**, Schrödinger's time-dependent equation reduces to

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x), \tag{7.30}$$

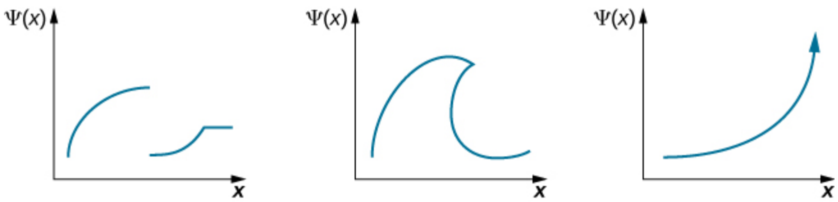
where E is the total energy of the particle (a real number). This equation is called **Schrödinger's time-independent equation**. Notice that we use "big psi" (Ψ) for the time-dependent wave function and "little psi" (ψ) for the time-independent wave function. The wave-function solution to this equation must be multiplied by the time-modulation factor to obtain the time-dependent wave function.

In the next sections, we solve Schrödinger's time-independent equation for three cases: a quantum particle in a box, a simple harmonic oscillator, and a quantum barrier. These cases provide important lessons that can be used to solve more complicated systems. The time-independent wave function $\psi(x)$ solutions must satisfy three conditions:

- $\psi(x)$ must be a continuous function.
- The first derivative of $\psi(x)$ with respect to space, $d\psi(x)/dx$, must be continuous, unless $V(x) = \infty$.
- $\psi(x)$ must not diverge ("blow up") at $x = \pm\infty$.

The first condition avoids sudden jumps or gaps in the wave function. The second condition requires the wave function to be smooth at all points, except in special cases. (In a more advanced course on quantum mechanics, for example, potential spikes of infinite depth and height are used to model solids). The third condition requires the wave function be normalizable. This third condition follows from Born's interpretation of quantum mechanics. It ensures that $|\psi(x)|^2$ is a finite number so we can use it to calculate probabilities.

7.6 Check Your Understanding Which of the following wave functions is a valid wave-function solution for Schrödinger's equation?



7.4 | The Quantum Particle in a Box

Learning Objectives

By the end of this section, you will be able to:

- Describe how to set up a boundary-value problem for the stationary Schrödinger equation
- Explain why the energy of a quantum particle in a box is quantized
- Describe the physical meaning of stationary solutions to Schrödinger's equation and the connection of these solutions with time-dependent quantum states
- Explain the physical meaning of Bohr's correspondence principle

In this section, we apply Schrödinger's equation to a particle bound to a one-dimensional box. This special case provides lessons for understanding quantum mechanics in more complex systems. The energy of the particle is quantized as a consequence of a standing wave condition inside the box.

Consider a particle of mass m that is allowed to move only along the x-direction and its motion is confined to the region between hard and rigid walls located at $x = 0$ and at $x = L$ (**Figure 7.10**). Between the walls, the particle moves freely. This physical situation is called the **infinite square well**, described by the potential energy function

$$U(x) = \begin{cases} 0, & 0 \le x \le L, \\ \infty, & \text{otherwise.} \end{cases} \tag{7.31}$$

Combining this equation with Schrödinger's time-independent wave equation gives

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x), \text{ for } 0 \le x \le L \tag{7.32}$$

where E is the total energy of the particle. What types of solutions do we expect? The energy of the particle is a positive number, so if the value of the wave function is positive (right side of the equation), the curvature of the wave function is negative, or concave down (left side of the equation). Similarly, if the value of the wave function is negative (right side of the equation), the curvature of the wave function is positive or concave up (left side of equation). This condition is met by an oscillating wave function, such as a sine or cosine wave. Since these waves are confined to the box, we envision standing waves with fixed endpoints at $x = 0$ and $x = L$.

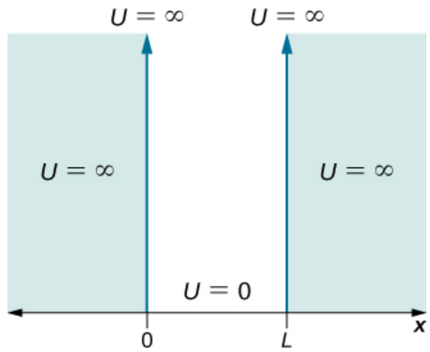


Figure 7.10 The potential energy function that confines the particle in a one-dimensional box.

Solutions $\psi(x)$ to this equation have a probabilistic interpretation. In particular, the square $|\psi(x)|^2$ represents the probability density of finding the particle at a particular location *x*. This function must be integrated to determine the probability of finding the particle in some interval of space. We are therefore looking for a normalizable solution that satisfies the following normalization condition:

$$\int_0^L dx|\psi(x)|^2 = 1. \tag{7.33}$$

The walls are rigid and impenetrable, which means that the particle is never found beyond the wall. Mathematically, this means that the solution must vanish at the walls:

$$\psi(0) = \psi(L) = 0. \tag{7.34}$$

We expect oscillating solutions, so the most general solution to this equation is

$$\psi_k(x) = A_k \cos kx + B_k \sin kx \tag{7.35}$$

where *k* is the wave number, and A_k and B_k are constants. Applying the boundary condition expressed by **Equation 7.34** gives

$$\psi_k(0) = A_k \cos(k \cdot 0) + B_k \sin(k \cdot 0) = A_k = 0. \tag{7.36}$$

Because we have $A_k = 0$, the solution must be

$$\psi_k(x) = B_k \sin kx. \tag{7.37}$$

If B_k is zero, $\psi_k(x) = 0$ for all values of x and the normalization condition, **Equation 7.33**, cannot be satisfied. Assuming $B_k \neq 0$, **Equation 7.34** for $x = L$ then gives

$$0 = B_k \sin(kL) \Rightarrow \sin(kL) = 0 \Rightarrow kL = n\pi,\ n = 1,\ 2,\ 3,... \tag{7.38}$$

We discard the $n = 0$ solution because $\psi(x)$ for this quantum number would be zero everywhere—an un-normalizable and therefore unphysical solution. Substituting **Equation 7.37** into **Equation 7.32** gives

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\big(B_k \sin(kx)\big) = E\big(B_k \sin(kx)\big). \tag{7.39}$$

Computing these derivatives leads to

$$E = E_k = \frac{\hbar^2 k^2}{2m}. \tag{7.40}$$

According to de Broglie, $p = \hbar k$, so this expression implies that the total energy is equal to the kinetic energy, consistent with our assumption that the "particle moves freely." Combining the results of **Equation 7.38** and **Equation 7.40** gives

$$E_n = n^2 \frac{\pi^2 \hbar^2}{2mL^2},\ n = 1,\ 2,\ 3,\ \ldots \tag{7.41}$$

Strange! A particle bound to a one-dimensional box can only have certain discrete (quantized) values of energy. Further, the particle cannot have a zero kinetic energy—it is impossible for a particle bound to a box to be "at rest."

To evaluate the allowed wave functions that correspond to these energies, we must find the normalization constant B_n. We impose the normalization condition **Equation 7.33** on the wave function

$$\psi_n(x) = B_n \sin n\pi x/L \tag{7.42}$$

$$1 = \int_0^L dx |\psi_n(x)|^2 = \int_0^L dx B_n^2 \sin^2 \frac{n\pi}{L} x = B_n^2 \int_0^L dx \sin^2 \frac{n\pi}{L} x = B_n^2 \frac{L}{2} \Rightarrow B_n = \sqrt{\frac{2}{L}}.$$

Hence, the wave functions that correspond to the energy values given in **Equation 7.41** are

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L},\ n = 1,\ 2,\ 3,\ \ldots \tag{7.43}$$

For the lowest energy state or **ground state energy**, we have

$$E_1 = \frac{\pi^2 \hbar^2}{2mL^2},\quad \psi_1(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right). \tag{7.44}$$

All other energy states can be expressed as

$$E_n = n^2 E_1,\ \psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right). \tag{7.45}$$

The index n is called the **energy quantum number** or **principal quantum number**. The state for $n = 2$ is the first excited state, the state for $n = 3$ is the second excited state, and so on. The first three quantum states (for $n = 1,\ 2,$ and 3) of a particle in a box are shown in **Figure 7.11**.

The wave functions in **Equation 7.45** are sometimes referred to as the "states of definite energy." Particles in these states are said to occupy **energy levels**, which are represented by the horizontal lines in **Figure 7.11**. Energy levels are analogous to rungs of a ladder that the particle can "climb" as it gains or loses energy.

The wave functions in **Equation 7.45** are also called **stationary states** and **standing wave states**. These functions are "stationary," because their probability density functions, $|\Psi(x,\ t)|^2$, do not vary in time, and "standing waves" because their real and imaginary parts oscillate up and down like a standing wave—like a rope waving between two children on a playground. Stationary states are states of definite energy [**Equation 7.45**], but linear combinations of these states, such as $\psi(x) = a\psi_1 + b\psi_2$ (also solutions to Schrödinger's equation) are states of mixed energy.

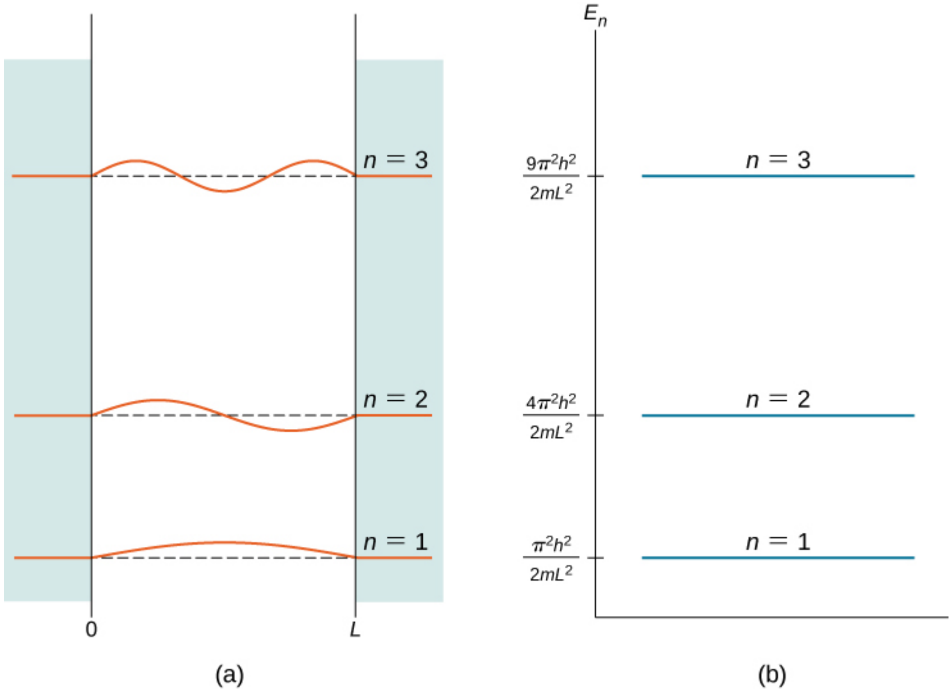


Figure 7.11 The first three quantum states of a quantum particle in a box for principal quantum numbers $n = 1,\ 2,\ \text{and}\ 3$: (a) standing wave solutions and (b) allowed energy states.

Energy quantization is a consequence of the boundary conditions. If the particle is not confined to a box but wanders freely, the allowed energies are continuous. However, in this case, only certain energies $(E_1,\ 4E_1,\ 9E_1,\ \ldots)$ are allowed. The energy difference between adjacent energy levels is given by

$$\Delta E_{n+1,\,n} = E_{n+1} - E_n = (n+1)^2 E_1 - n^2 E_1 = (2n+1)E_1. \qquad (7.46)$$

Conservation of energy demands that if the energy of the system changes, the energy difference is carried in some other form of energy. For the special case of a charged particle confined to a small volume (for example, in an atom), energy changes are often carried away by photons. The frequencies of the emitted photons give us information about the energy differences (spacings) of the system and the volume of containment—the size of the "box" [see **Equation 7.44**].

Example 7.8

A Simple Model of the Nucleus

Suppose a proton is confined to a box of width $L = 1.00 \times 10^{-14}\,\text{m}$ (a typical nuclear radius). What are the energies of the ground and the first excited states? If the proton makes a transition from the first excited state to the ground state, what are the energy and the frequency of the emitted photon?

Strategy

If we assume that the proton confined in the nucleus can be modeled as a quantum particle in a box, all we need to do is to use **Equation 7.41** to find its energies E_1 and E_2. The mass of a proton is $m = 1.76 \times 10^{-27}$ kg. The emitted photon carries away the energy difference $\Delta E = E_2 - E_1$. We can use the relation $E_f = hf$ to find its frequency *f*.

Solution

The ground state:

$$E_1 = \frac{\pi^2 \hbar^2}{2mL^2} = \frac{\pi^2 (1.05 \times 10^{-34}\,\text{J}\cdot\text{s})^2}{2(1.67 \times 10^{-27}\,\text{kg})\,(1.00 \times 10^{-14}\,\text{m})^2} = 3.28 \times 10^{-13}\,\text{J} = 2.05\,\text{MeV}.$$

The first excited state: $E_2 = 2^2 E_1 = 4(2.05\text{ MeV}) = 8.20\text{ MeV}$.

The energy of the emitted photon is $E_f = \Delta E = E_2 - E_1 = 8.20\text{ MeV} - 2.05\text{ MeV} = 6.15\text{ MeV}$.

The frequency of the emitted photon is

$$f = \frac{E_f}{h} = \frac{6.15\text{ MeV}}{4.14 \times 10^{-21}\text{ MeV} \cdot \text{s}} = 1.49 \times 10^{21}\text{ Hz}.$$

Significance

This is the typical frequency of a gamma ray emitted by a nucleus. The energy of this photon is about 10 million times greater than that of a visible light photon.

The expectation value of the position for a particle in a box is given by

$$\langle x \rangle = \int_0^L dx \psi_n^*(x) x \psi_n(x) = \int_0^L dx x |\psi_n^*(x)|^2 = \int_0^L dx x \frac{2}{L} \sin^2 \frac{n\pi x}{L} = \frac{L}{2}. \quad (7.47)$$

We can also find the expectation value of the momentum or average momentum of a large number of particles in a given state:

$$\begin{aligned} \langle p \rangle &= \int_0^L dx \psi_n^*(x) \left[-i\hbar \frac{d}{dx} \psi_n(x) \right] \\ &= -i\hbar \int_0^L dx \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \left[\frac{d}{dx} \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \right] = -i \frac{2\hbar}{L} \int_0^L dx \sin \frac{n\pi x}{L} \left[\frac{n\pi}{L} \cos \frac{n\pi x}{L} \right] \\ &= -i \frac{2n\pi\hbar}{L^2} \int_0^L dx \frac{1}{2} \sin \frac{2n\pi x}{L} = -i \frac{n\pi\hbar}{L^2} \frac{L}{2n\pi} \int_0^{2\pi n} d\varphi \sin \varphi = -i \frac{\hbar}{2L} \cdot 0 = 0. \end{aligned} \quad (7.48)$$

Thus, for a particle in a state of definite energy, the average position is in the middle of the box and the average momentum of the particle is zero—as it would also be for a classical particle. Note that while the minimum energy of a classical particle can be zero (the particle can be at rest in the middle of the box), the minimum energy of a quantum particle is nonzero and given by **Equation 7.44**. The average particle energy in the *nth* quantum state—its expectation value of energy—is

$$E_n = \langle E \rangle = n^2 \frac{\pi^2 \hbar^2}{2m}. \quad (7.49)$$

The result is not surprising because the standing wave state is a state of definite energy. Any energy measurement of this system must return a value equal to one of these allowed energies.

Our analysis of the quantum particle in a box would not be complete without discussing Bohr's correspondence principle. This principle states that for large quantum numbers, the laws of quantum physics must give identical results as the laws of classical physics. To illustrate how this principle works for a quantum particle in a box, we plot the probability density distribution

$$\left| \psi_n(x) \right|^2 = \frac{2}{L} \sin^2(n\pi x/L) \quad (7.50)$$

for finding the particle around location *x* between the walls when the particle is in quantum state ψ_n. **Figure 7.12** shows these probability distributions for the ground state, for the first excited state, and for a highly excited state that corresponds to a large quantum number. We see from these plots that when a quantum particle is in the ground state, it is most likely to be found around the middle of the box, where the probability distribution has the largest value. This is not so when the particle is in the first excited state because now the probability distribution has the zero value in the middle of the box, so there is no chance of finding the particle there. When a quantum particle is in the first excited state, the probability distribution has two maxima, and the best chance of finding the particle is at positions close to the locations of these maxima. This quantum picture is unlike the classical picture.

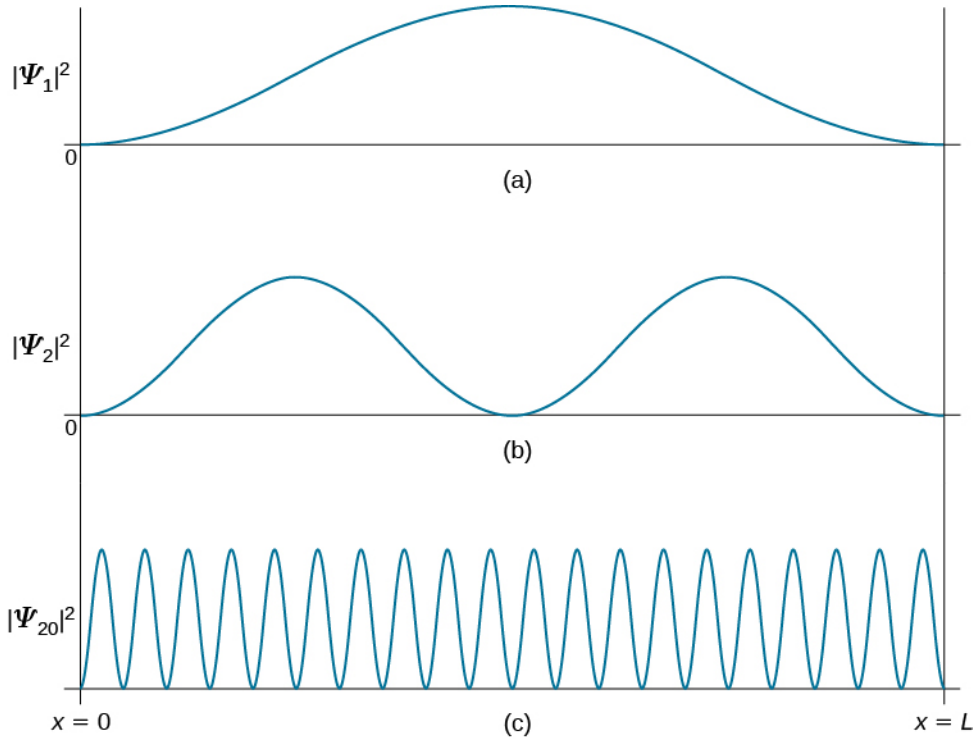


Figure 7.12 The probability density distribution $|\psi_n(x)|^2$ for a quantum particle in a box for: (a) the ground state, $n = 1$; (b) the first excited state, $n = 2$; and, (c) the nineteenth excited state, $n = 20$.

The probability density of finding a classical particle between x and $x + \Delta x$ depends on how much time Δt the particle spends in this region. Assuming that its speed u is constant, this time is $\Delta t = \Delta x/u$, which is also constant for any location between the walls. Therefore, the probability density of finding the classical particle at x is uniform throughout the box, and there is no preferable location for finding a classical particle. This classical picture is matched in the limit of large quantum numbers. For example, when a quantum particle is in a highly excited state, shown in **Figure 7.12**, the probability density is characterized by rapid fluctuations and then the probability of finding the quantum particle in the interval Δx does not depend on where this interval is located between the walls.

Example 7.9

A Classical Particle in a Box

A small 0.40-kg cart is moving back and forth along an air track between two bumpers located 2.0 m apart. We assume no friction; collisions with the bumpers are perfectly elastic so that between the bumpers, the car maintains a constant speed of 0.50 m/s. Treating the cart as a quantum particle, estimate the value of the principal quantum number that corresponds to its classical energy.

Strategy

We find the kinetic energy K of the cart and its ground state energy E_1 as though it were a quantum particle. The energy of the cart is completely kinetic, so $K = n^2 E_1$ (**Equation 7.45**). Solving for n gives $n = (K/E_1)^{1/2}$.

Solution

The kinetic energy of the cart is

$$K = \frac{1}{2}mu^2 = \frac{1}{2}(0.40\ \text{kg})(0.50\ \text{m/s})^2 = 0.050\ \text{J}.$$

The ground state of the cart, treated as a quantum particle, is

$$E_1 = \frac{\pi^2 \hbar^2}{2mL^2} = \frac{\pi^2 (1.05 \times 10^{-34}\,\mathrm{J \cdot s})^2}{2(0.40\,\mathrm{kg})(2.0\,\mathrm{m})^2} = 1.700 \times 10^{-68}\,\mathrm{J}.$$

Therefore, $n = (K/E_1)^{1/2} = (0.050/1.700 \times 10^{-68})^{1/2} = 1.2 \times 10^{33}$.

Significance

We see from this example that the energy of a classical system is characterized by a very large quantum number. Bohr's correspondence principle concerns this kind of situation. We can apply the formalism of quantum mechanics to any kind of system, quantum or classical, and the results are correct in each case. In the limit of high quantum numbers, there is no advantage in using quantum formalism because we can obtain the same results with the less complicated formalism of classical mechanics. However, we cannot apply classical formalism to a quantum system in a low-number energy state.

7.7 Check Your Understanding (a) Consider an infinite square well with wall boundaries $x = 0$ and $x = L$. What is the probability of finding a quantum particle in its ground state somewhere between $x = 0$ and $x = L/4$? (b) Repeat question (a) for a classical particle.

Having found the stationary states $\psi_n(x)$ and the energies E_n by solving the time-independent Schrödinger equation **Equation 7.32**, we use **Equation 7.28** to write wave functions $\Psi_n(x, t)$ that are solutions of the time-dependent Schrödinger's equation given by **Equation 7.23**. For a particle in a box this gives

$$\Psi_n(x, t) = e^{-i\omega_n t}\psi_n(x) = \sqrt{\frac{2}{L}} e^{-iE_n t/\hbar} \sin\frac{n\pi x}{L}, \ n = 1, 2, 3, \ldots \tag{7.51}$$

where the energies are given by **Equation 7.41**.

The quantum particle in a box model has practical applications in a relatively newly emerged field of optoelectronics, which deals with devices that convert electrical signals into optical signals. This model also deals with nanoscale physical phenomena, such as a nanoparticle trapped in a low electric potential bounded by high-potential barriers.

7.5 | The Quantum Harmonic Oscillator

Learning Objectives

By the end of this section, you will be able to:

- Describe the model of the quantum harmonic oscillator
- Identify differences between the classical and quantum models of the harmonic oscillator
- Explain physical situations where the classical and the quantum models coincide

Oscillations are found throughout nature, in such things as electromagnetic waves, vibrating molecules, and the gentle back-and-forth sway of a tree branch. In previous chapters, we used Newtonian mechanics to study macroscopic oscillations, such as a block on a spring and a simple pendulum. In this chapter, we begin to study oscillating systems using quantum mechanics. We begin with a review of the classic harmonic oscillator.

The Classic Harmonic Oscillator

A simple harmonic oscillator is a particle or system that undergoes harmonic motion about an equilibrium position, such as an object with mass vibrating on a spring. In this section, we consider oscillations in one-dimension only. Suppose a mass moves back-and-forth along the

x-direction about the equilibrium position, $x = 0$. In classical mechanics, the particle moves in response to a linear restoring force given by $F_x = -kx$, where x is the displacement of the particle from its equilibrium position. The motion takes place between two turning points, $x = \pm A$, where A denotes the amplitude of the motion. The position of the object varies periodically in time with angular frequency $\omega = \sqrt{k/m}$, which depends on the mass *m* of the oscillator and on the force constant *k* of the net force, and can be written as

$$x(t) = A\cos(\omega t + \phi). \tag{7.52}$$

The total energy *E* of an oscillator is the sum of its kinetic energy $K = mu^2/2$ and the elastic potential energy of the force $U(x) = kx^2/2$,

$$E = \frac{1}{2}mu^2 + \frac{1}{2}kx^2. \tag{7.53}$$

At turning points $x = \pm A$, the speed of the oscillator is zero; therefore, at these points, the energy of oscillation is solely in the form of potential energy $E = kA^2/2$. The plot of the potential energy $U(x)$ of the oscillator versus its position *x* is a parabola (**Figure 7.13**). The potential-energy function is a quadratic function of *x*, measured with respect to the equilibrium position. On the same graph, we also plot the total energy *E* of the oscillator, as a horizontal line that intercepts the parabola at $x = \pm A$. Then the kinetic energy *K* is represented as the vertical distance between the line of total energy and the potential energy parabola.

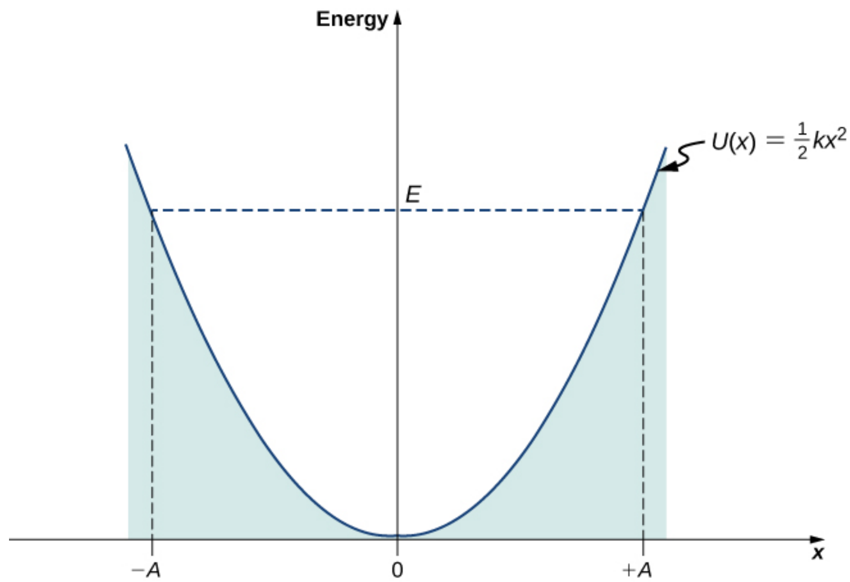


Figure 7.13 The potential energy well of a classical harmonic oscillator: The motion is confined between turning points at $x = -A$ and at $x = +A$. The energy of oscillations is $E = kA^2/2$.

In this plot, the motion of a classical oscillator is confined to the region where its kinetic energy is nonnegative, which is what the energy relation **Equation 7.53** says. Physically, it means that a classical oscillator can never be found beyond its turning points, and its energy depends only on how far the turning points are from its equilibrium position. The energy of a classical oscillator changes in a continuous way. The lowest energy that a classical oscillator may have is zero, which corresponds to a situation where an object is at rest at its equilibrium position. The zero-energy state of a classical oscillator simply means no oscillations and no motion at all (a classical particle sitting at the bottom of the potential well in **Figure 7.13**). When an object oscillates, no matter how big or small its energy may be, it spends the longest time near the turning points, because this is where it slows down and reverses its direction of motion. Therefore, the probability of finding a classical oscillator between the turning points is highest near the turning points and lowest at the equilibrium position. (Note that this is not a statement of preference of the object to go to lower energy. It is a statement about how quickly the object moves through various regions.)

The Quantum Harmonic Oscillator

One problem with this classical formulation is that it is not general. We cannot use it, for example, to describe vibrations of diatomic molecules, where quantum effects are important. A first step toward a quantum formulation is to use the classical expression $k = m\omega^2$ to limit mention of a "spring" constant between the atoms. In this way the potential energy function can be written in a more general form,

$$U(x) = \frac{1}{2}m\omega^2 x^2. \tag{7.54}$$

Combining this expression with the time-independent Schrödinger equation gives

$$-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi(x) = E\psi(x). \tag{7.55}$$

To solve **Equation 7.55**—that is, to find the allowed energies E and their corresponding wave functions $\psi(x)$ —we require the wave functions to be symmetric about $x = 0$ (the bottom of the potential well) and to be normalizable. These conditions ensure that the probability density $|\psi(x)|^2$ must be finite when integrated over the entire range of x from $-\infty$ to $+\infty$. How to solve **Equation 7.55** is the subject of a more advanced course in quantum mechanics; here, we simply cite the results. The allowed energies are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega = \frac{2n+1}{2}\hbar\omega, \quad n = 0, 1, 2, 3, ... \tag{7.56}$$

The wave functions that correspond to these energies (the stationary states or states of definite energy) are

$$\psi_n(x) = N_n e^{-\beta^2 x^2/2} H_n(\beta x), \quad n = 0, 1, 2, 3, ... \tag{7.57}$$

where $\beta = \sqrt{m\omega/\hbar}$, N_n is the normalization constant, and $H_n(y)$ is a polynomial of degree n called a *Hermite polynomial*. The first four Hermite polynomials are

$$H_0(y) = 1$$
$$H_1(y) = 2y$$
$$H_2(y) = 4y^2 - 2$$
$$H_3(y) = 8y^3 - 12y.$$

A few sample wave functions are given in **Figure 7.14**. As the value of the principal number increases, the solutions alternate between even functions and odd functions about $x = 0$.

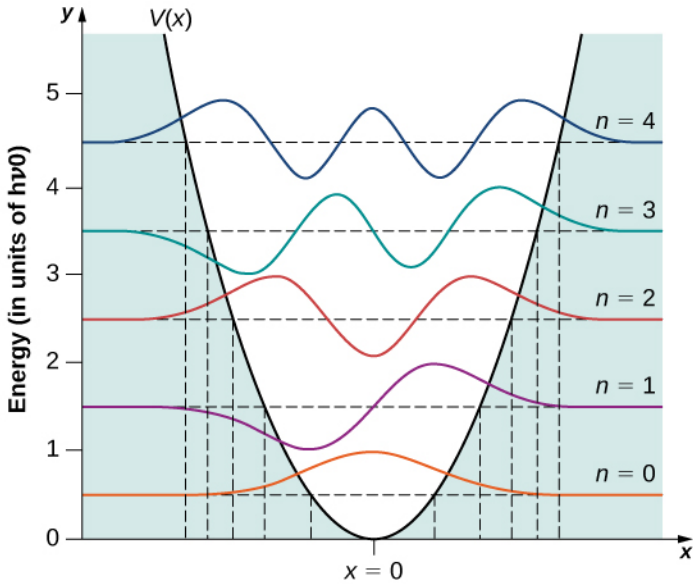


Figure 7.14 The first five wave functions of the quantum harmonic oscillator. The classical limits of the oscillator's motion are indicated by vertical lines, corresponding to the classical turning points at $x = \pm A$ of a classical particle with the same energy as the energy of a quantum oscillator in the state indicated in the figure.

Example 7.10

Classical Region of Harmonic Oscillations

Find the amplitude A of oscillations for a classical oscillator with energy equal to the energy of a quantum oscillator in the quantum state n.

Strategy

To determine the amplitude A, we set the classical energy $E = kx^2/2 = m\omega^2 A^2/2$ equal to E_n given by **Equation 7.56**.

Solution

We obtain

$$E_n = m\omega^2 A_n^2/2 \;\; \Rightarrow \;\; A_n = \sqrt{\frac{2}{m\omega^2}E_n} = \sqrt{\frac{2}{m\omega^2}\frac{2n+1}{2}\hbar\omega} = \sqrt{(2n+1)\frac{\hbar}{m\omega}}.$$

Significance

As the quantum number n increases, the energy of the oscillator and therefore the amplitude of oscillation increases (for a fixed natural angular frequency. For large n, the amplitude is approximately proportional to the square root of the quantum number.

Several interesting features appear in this solution. Unlike a classical oscillator, the measured energies of a quantum oscillator can have only energy values given by **Equation 7.56**. Moreover, unlike the case for a quantum particle in a box, the allowable energy levels are evenly spaced,

$$\Delta E = E_{n+1} - E_n = \frac{2(n+1)+1}{2}\hbar\omega - \frac{2n+1}{2}\hbar\omega = \hbar\omega = hf. \tag{7.58}$$

When a particle bound to such a system makes a transition from a higher-energy state to a lower-energy state, the smallest-energy quantum carried by the emitted photon is necessarily hf. Similarly, when the particle makes a transition from a lower-energy state to a higher-energy state, the smallest-energy quantum that can be absorbed by the particle is hf. A quantum oscillator can absorb or emit energy only in multiples of this smallest-energy quantum. This is consistent with Planck's hypothesis for the energy exchanges between radiation and the cavity walls in the blackbody radiation problem.

Example 7.11

Vibrational Energies of the Hydrogen Chloride Molecule

The HCl diatomic molecule consists of one chlorine atom and one hydrogen atom. Because the chlorine atom is 35 times more massive than the hydrogen atom, the vibrations of the HCl molecule can be quite well approximated by assuming that the Cl atom is motionless and the H atom performs harmonic oscillations due to an elastic molecular force modeled by Hooke's law. The infrared vibrational spectrum measured for hydrogen chloride has the lowest-frequency line centered at $f = 8.88 \times 10^{13}$ Hz. What is the spacing between the vibrational energies of this molecule? What is the force constant *k* of the atomic bond in the HCl molecule?

Strategy

The lowest-frequency line corresponds to the emission of lowest-frequency photons. These photons are emitted when the molecule makes a transition between two adjacent vibrational energy levels. Assuming that energy levels are equally spaced, we use **Equation 7.58** to estimate the spacing. The molecule is well approximated by treating the Cl atom as being infinitely heavy and the H atom as the mass *m* that performs the oscillations. Treating this molecular system as a classical oscillator, the force constant is found from the classical relation $k = m\omega^2$.

Solution

The energy spacing is

$$\Delta E = hf = (4.14 \times 10^{-15}\,\text{eV} \cdot \text{s})(8.88 \times 10^{13}\ \text{Hz}) = 0.368\,\text{eV}.$$

The force constant is

$$k = m\omega^2 = m(2\pi f)^2 = (1.67 \times 10^{-27}\ \text{kg})(2\pi \times 8.88 \times 10^{13}\ \text{Hz})^2 = 520\,\text{N/m}.$$

Significance

The force between atoms in an HCl molecule is surprisingly strong. The typical energy released in energy transitions between vibrational levels is in the infrared range. As we will see later, transitions in between vibrational energy levels of a diatomic molecule often accompany transitions between rotational energy levels.

7.8 Check Your Understanding The vibrational frequency of the hydrogen iodide HI diatomic molecule is 6.69×10^{13} Hz. (a) What is the force constant of the molecular bond between the hydrogen and the iodine atoms? (b) What is the energy of the emitted photon when this molecule makes a transition between adjacent vibrational energy levels?

The quantum oscillator differs from the classic oscillator in three ways:

First, the ground state of a quantum oscillator is $E_0 = \hbar\omega/2,$ not zero. In the classical view, the lowest energy is zero. The nonexistence of a zero-energy state is common for all quantum-mechanical systems because of omnipresent fluctuations that are a consequence of the Heisenberg uncertainty principle. If a quantum particle sat motionless at the bottom of the potential well, its momentum as well as its position would have to be simultaneously exact, which would violate the Heisenberg uncertainty principle. Therefore, the lowest-energy state must be characterized by uncertainties in momentum and in position, so the ground state of a quantum particle must lie above the bottom of the potential well.

Second, a particle in a quantum harmonic oscillator potential can be found with nonzero probability outside the interval $-A \le x \le +A$. In a classic formulation of the problem, the particle would not have any energy to be in this region. The probability of finding a ground-state quantum particle in the classically forbidden region is about 16%.

Third, the probability density distributions $|\psi_n(x)|^2$ for a quantum oscillator in the ground low-energy state, $\psi_0(x)$, is largest at the middle of the well $(x = 0)$. For the particle to be found with greatest probability at the center of the well, we expect that the particle spends the most time there as it oscillates. This is opposite to the behavior of a classical oscillator, in which the particle spends most of its time moving with relative small speeds near the turning points.

7.9 Check Your Understanding Find the expectation value of the position for a particle in the ground state of a harmonic oscillator using symmetry.

Quantum probability density distributions change in character for excited states, becoming more like the classical distribution when the quantum number gets higher. We observe this change already for the first excited state of a quantum oscillator because the distribution $|\psi_1(x)|^2$ peaks up around the turning points and vanishes at the equilibrium position, as seen in **Figure 7.13**. In accordance with Bohr's correspondence principle, in the limit of high quantum numbers, the quantum description of a harmonic oscillator converges to the classical description, which is illustrated in **Figure 7.15**. The classical probability density distribution corresponding to the quantum energy of the $n = 12$ state is a reasonably good approximation of the quantum probability distribution for a quantum oscillator in this excited state. This agreement becomes increasingly better for highly excited states.

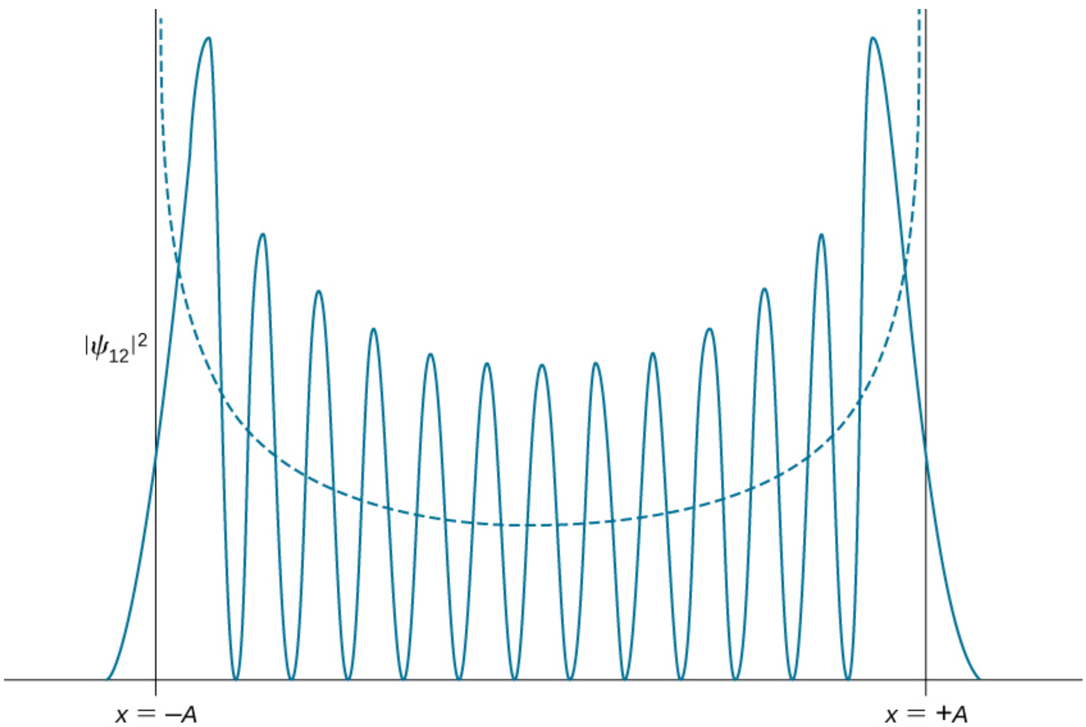


Figure 7.15 The probability density distribution for finding the quantum harmonic oscillator in its $n = 12$ quantum state. The dashed curve shows the probability density distribution of a classical oscillator with the same energy.

7.6 | The Quantum Tunneling of Particles through Potential Barriers

Learning Objectives

By the end of this section, you will be able to:

- Describe how a quantum particle may tunnel across a potential barrier
- Identify important physical parameters that affect the tunneling probability
- Identify the physical phenomena where quantum tunneling is observed
- Explain how quantum tunneling is utilized in modern technologies

Quantum tunneling is a phenomenon in which particles penetrate a potential energy barrier with a height greater than the total energy of the particles. The phenomenon is interesting and important because it violates the principles of classical mechanics. Quantum tunneling is important in models of the Sun and has a wide range of applications, such as the scanning tunneling microscope and the tunnel diode.

Tunneling and Potential Energy

To illustrate quantum tunneling, consider a ball rolling along a surface with a kinetic energy of 100 J. As the ball rolls, it encounters a hill. The potential energy of the ball placed atop the hill is 10 J. Therefore, the ball (with 100 J of kinetic energy) easily rolls over the hill and continues on. In classical mechanics, the probability that the ball passes over the hill is exactly 1—it makes it over every time. If, however, the height of the hill is increased—a ball placed atop the hill has a potential energy of 200 J—the ball proceeds only part of the way up the hill, stops, and returns in the direction it came. The total energy of the ball is converted entirely into potential energy before it can reach the top of the hill. We do not expect, even after repeated attempts, for the 100-J ball to ever be found beyond the hill. Therefore, the probability that the ball passes over the hill is exactly 0, and probability it is turned back or “reflected” by the hill is exactly 1. The ball *never* makes it over the hill. The existence of the ball beyond the hill is an impossibility or “energetically forbidden.”

However, according to quantum mechanics, the ball has a wave function and this function is defined over all space. The wave function may be highly localized, but there is always a chance that as the ball encounters the hill, the ball will suddenly be found beyond it. Indeed, this probability is appreciable if the “wave packet” of the ball is wider than the barrier.

View this **interactive simulation (https://openstaxcollege.org/l/21intquatanvid)** for a simulation of tunneling.

In the language of quantum mechanics, the hill is characterized by a **potential barrier**. A finite-height square barrier is described by the following potential-energy function:

$$U(x) = \begin{cases} 0, & \text{when } x < 0 \\ U_0, & \text{when } 0 \le x \le L \\ 0, & \text{when } x > L. \end{cases} \tag{7.59}$$

The potential barrier is illustrated in **Figure 7.16**. When the height U_0 of the barrier is infinite, the wave packet representing an incident quantum particle is unable to penetrate it, and the quantum particle bounces back from the barrier boundary, just like a classical particle. When the width L of the barrier is infinite and its height is finite, a part of the wave packet representing an incident quantum particle can filter through the barrier boundary and eventually perish after traveling some distance inside the barrier.

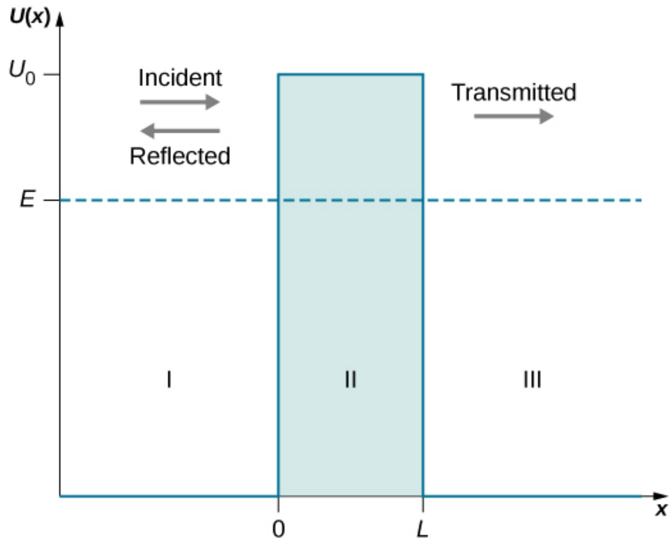


Figure 7.16 A potential energy barrier of height U_0 creates three physical regions with three different wave behaviors. In region I where $x < 0$, an incident wave packet (incident particle) moves in a potential-free zone and coexists with a reflected wave packet (reflected particle). In region II, a part of the incident wave that has not been reflected at $x = 0$ moves as a transmitted wave in a constant potential $U(x) = +U_0$ and tunnels through to region III at $x = L$. In region III for $x > L$, a wave packet (transmitted particle) that has tunneled through the potential barrier moves as a free particle in potential-free zone. The energy E of the incident particle is indicated by the horizontal line.

When both the width L and the height U_0 are finite, a part of the quantum wave packet incident on one side of the barrier can penetrate the barrier boundary and continue its motion inside the barrier, where it is gradually attenuated on its way to the other side. A part of the incident quantum wave packet eventually emerges on the other side of the barrier in the form of the transmitted wave packet that tunneled through the barrier. How much of the incident wave can tunnel through a barrier depends on the barrier width L and its height U_0, and on the energy E of the quantum particle incident on the barrier. This is the physics of tunneling.

Barrier penetration by quantum wave functions was first analyzed theoretically by Friedrich Hund in 1927, shortly after Schrödinger published the equation that bears his name. A year later, George Gamow used the formalism of quantum mechanics to explain the radioactive α-decay of atomic nuclei as a quantum-tunneling phenomenon. The invention of the tunnel diode in 1957 made it clear that quantum tunneling is important to the semiconductor industry. In modern nanotechnologies, individual atoms are manipulated using a knowledge of quantum tunneling.

Tunneling and the Wave Function

Suppose a uniform and time-independent beam of electrons or other quantum particles with energy E traveling along the x-axis (in the positive direction to the right) encounters a potential barrier described by **Equation 7.59**. The question is: What is the probability that an individual particle in the beam will tunnel through the potential barrier? The answer can be found by solving the boundary-value problem for the time-independent Schrödinger equation for a particle in the beam. The general form of this equation is given by **Equation 7.60**, which we reproduce here:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x), \text{ where } -\infty < x < +\infty. \quad (7.60)$$

In **Equation 7.60**, the potential function $U(x)$ is defined by **Equation 7.59**. We assume that the given energy E of the incoming particle is smaller than the height U_0 of the potential barrier, $E < U_0$, because this is the interesting physical case. Knowing the energy E of the incoming particle, our task is to solve **Equation 7.60** for a function $\psi(x)$ that is continuous and has continuous first derivatives for all x. In other words, we are looking for a “smooth-looking” solution (because this is how wave functions look) that can be given a probabilistic interpretation so that $|\psi(x)|^2 = \psi^*(x)\psi(x)$ is the probability density.

We divide the real axis into three regions with the boundaries defined by the potential function in **Equation 7.59** (illustrated in **Figure 7.16**) and transcribe **Equation 7.60** for each region. Denoting by $\psi_{\text{I}}(x)$ the solution in region I for $x < 0$, by $\psi_{\text{II}}(x)$ the solution in region II for $0 \le x \le L$, and by $\psi_{\text{III}}(x)$ the solution in region III for $x > L$, the stationary Schrӧdinger equation has the following forms in these three regions:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{\text{I}}(x)}{dx^2} = E\psi_{\text{I}}(x),\ \text{in region I:} -\infty < x < 0, \tag{7.61}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{\text{II}}(x)}{dx^2} + U_0\psi_{\text{II}}(x) = E\psi_{\text{II}}(x),\ \text{in region II:}\ 0 \le x \le L, \tag{7.62}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{\text{III}}(x)}{dx^2} = E\psi_{\text{III}}(x),\ \text{in region III:}\ L < x < +\infty. \tag{7.63}$$

The continuity condition at region boundaries requires that:

$$\psi_{\text{I}}(0) = \psi_{\text{II}}(0),\ \text{at the boundary between regions I and II and} \tag{7.64}$$

and

$$\psi_{\text{II}}(L) = \psi_{\text{III}}(L),\ \text{at the boundary between regions II and III.} \tag{7.65}$$

The “smoothness” condition requires the first derivative of the solution be continuous at region boundaries:

$$\left.\frac{d\psi_{\text{I}}(x)}{dx}\right|_{x=0} = \left.\frac{d\psi_{\text{II}}(x)}{dx}\right|_{x=0},\ \text{at the boundary between regions I and II;} \tag{7.66}$$

and

$$\left.\frac{d\psi_{\text{II}}(x)}{dx}\right|_{x=L} = \left.\frac{d\psi_{\text{III}}(x)}{dx}\right|_{x=L},\ \text{at the boundary between regions II and III.} \tag{7.67}$$

In what follows, we find the functions $\psi_{\text{I}}(x)$, $\psi_{\text{II}}(x)$, and $\psi_{\text{III}}(x)$.

We can easily verify (by substituting into the original equation and differentiating) that in regions I and III, the solutions must be in the following general forms:

$$\psi_{\text{I}}(x) = Ae^{+ikx} + Be^{-ikx} \tag{7.68}$$

$$\psi_{\text{III}}(x) = Fe^{+ikx} + Ge^{-ikx} \tag{7.69}$$

where $k = \sqrt{2mE}/\hbar$ is a wave number and the complex exponent denotes oscillations,

$$e^{\pm ikx} = \cos kx \pm i \sin kx. \tag{7.70}$$

The constants A, B, F, and G in **Equation 7.68** and **Equation 7.69** may be complex. These solutions are illustrated in **Figure 7.16**. In region I, there are two waves—one is incident (moving to the right) and one is reflected (moving to the left)—so none of the constants A and B in **Equation 7.68** may vanish. In region III, there is only one wave (moving to the right), which is the transmitted wave, so the constant G must be zero in **Equation 7.69**, $G = 0$. We can write explicitly that the incident wave is $\psi_{\text{in}}(x) = Ae^{+ikx}$ and that the reflected wave is $\psi_{\text{ref}}(x) = Be^{-ikx}$, and that the transmitted wave is $\psi_{\text{tra}}(x) = Fe^{+ikx}$. The amplitude of the incident wave is

$$\left|\psi_{\text{in}}(x)\right|^2 = \psi_{\text{in}}^*(x)\psi_{\text{in}}(x) = \left(Ae^{+ikx}\right)^* Ae^{+ikx} = A^*\, e^{-ikx} Ae^{+ikx} = A^*\, A = \left|A\right|^2.$$

Similarly, the amplitude of the reflected wave is $|\psi_{\text{ref}}(x)|^2 = |B|^2$ and the amplitude of the transmitted wave is $|\psi_{\text{tra}}(x)|^2 = |F|^2$. We know from the theory of waves that the square of the wave amplitude is directly proportional to the wave intensity. If we want to know how much of the incident wave tunnels through the barrier, we need to compute the square of the amplitude of the transmitted wave. The **transmission probability** or **tunneling probability** is the ratio of the transmitted intensity $(|F|^2)$ to the incident intensity $(|A|^2)$, written as

$$T(L, E) = \frac{|\psi_{\text{tra}}(x)|^2}{|\psi_{\text{in}}(x)|^2} = \frac{|F|^2}{|A|^2} = \left|\frac{F}{A}\right|^2 \quad (7.71)$$

where L is the width of the barrier and E is the total energy of the particle. This is the probability an individual particle in the incident beam will tunnel through the potential barrier. Intuitively, we understand that this probability must depend on the barrier height U_0.

In region II, the terms in equation **Equation 7.62** can be rearranged to

$$\frac{d^2\psi_{\text{II}}(x)}{dx^2} = \beta^2\psi_{\text{II}}(x) \quad (7.72)$$

where β^2 is positive because $U_0 > E$ and the parameter β is a real number,

$$\beta^2 = \frac{2m}{\hbar^2}(U_0 - E). \quad (7.73)$$

The general solution to **Equation 7.72** is not oscillatory (unlike in the other regions) and is in the form of exponentials that describe a gradual attenuation of $\psi_{\text{II}}(x)$,

$$\psi_{\text{II}}(x) = Ce^{-\beta x} + De^{+\beta x}. \quad (7.74)$$

The two types of solutions in the three regions are illustrated in **Figure 7.17**.

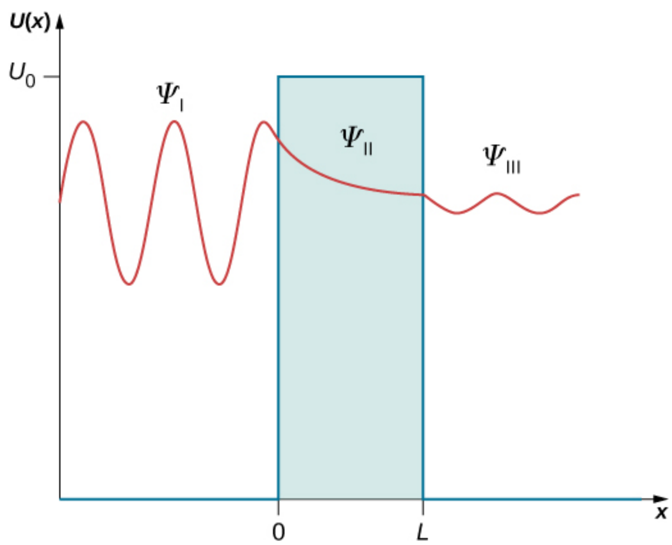


Figure 7.17 Three types of solutions to the stationary Schrödinger equation for the quantum-tunneling problem: Oscillatory behavior in regions I and III where a quantum particle moves freely, and exponential-decay behavior in region II (the barrier region) where the particle moves in the potential U_0.

Now we use the boundary conditions to find equations for the unknown constants. **Equation 7.68** and **Equation 7.74** are substituted into **Equation 7.64** to give

$$A + B = C + D. \tag{7.75}$$

Equation 7.74 and **Equation 7.69** are substituted into **Equation 7.65** to give

$$Ce^{-\beta L} + De^{+\beta L} = Fe^{+ikL}. \tag{7.76}$$

Similarly, we substitute **Equation 7.68** and **Equation 7.74** into **Equation 7.66**, differentiate, and obtain

$$-ik(A - B) = \beta(D - C). \tag{7.77}$$

Similarly, the boundary condition **Equation 7.67** reads explicitly

$$\beta\left(De^{+\beta L} - Ce^{-\beta L}\right) = -ikFe^{+ikL}. \tag{7.78}$$

We now have four equations for five unknown constants. However, because the quantity we are after is the transmission coefficient, defined in **Equation 7.71** by the fraction F/A, the number of equations is exactly right because when we divide each of the above equations by A, we end up having only four unknown fractions: B/A, C/A, D/A, and F/A, three of which can be eliminated to find F/A. The actual algebra that leads to expression for F/A is pretty lengthy, but it can be done either by hand or with a help of computer software. The end result is

$$\frac{F}{A} = \frac{e^{-ikL}}{\cosh(\beta L) + i(\gamma/2)\sinh(\beta L)}. \tag{7.79}$$

In deriving **Equation 7.79**, to avoid the clutter, we use the substitutions $\gamma \equiv \beta/k - k/\beta$,

$$\cosh y = \frac{e^y + e^{-y}}{2}, \text{ and } \sinh y = \frac{e^y - e^{-y}}{2}.$$

We substitute **Equation 7.79** into **Equation 7.71** and obtain the exact expression for the transmission coefficient for the barrier,

$$T(L, E) = \left(\frac{F}{A}\right)^* \frac{F}{A} = \frac{e^{+ikL}}{\cosh(\beta L) - i(\gamma/2)\sinh(\beta L)} \cdot \frac{e^{-ikL}}{\cosh(\beta L) + i(\gamma/2)\sinh(\beta L)}$$

or

$$T(L, E) = \frac{1}{\cosh^2(\beta L) + (\gamma/2)^2 \sinh^2(\beta L)} \tag{7.80}$$

where

$$\left(\frac{\gamma}{2}\right)^2 = \frac{1}{4}\left(\frac{1 - E/U_0}{E/U_0} + \frac{E/U_0}{1 - E/U_0} - 2\right).$$

For a wide and high barrier that transmits poorly, **Equation 7.80** can be approximated by

$$T(L, E) = 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right)e^{-2\beta L}. \tag{7.81}$$

Whether it is the exact expression **Equation 7.80** or the approximate expression **Equation 7.81**, we see that the tunneling effect very strongly depends on the width L of the potential barrier. In the laboratory, we can adjust both the potential height U_0 and the width L to design nano-devices with desirable transmission coefficients.

Example 7.12

Transmission Coefficient

Two copper nanowires are insulated by a copper oxide nano-layer that provides a 10.0-eV potential barrier. Estimate the tunneling probability between the nanowires by 7.00-eV electrons through a 5.00-nm thick oxide layer. What if the thickness of the layer were reduced to just 1.00 nm? What if the energy of electrons were increased to 9.00 eV?

Strategy

Treating the insulating oxide layer as a finite-height potential barrier, we use **Equation 7.81**. We identify $U_0 = 10.0\,\text{eV}$, $E_1 = 7.00\,\text{eV}$, $E_2 = 9.00\,\text{eV}$, $L_1 = 5.00\,\text{nm}$, and $L_2 = 1.00\,\text{nm}$. We use **Equation 7.73** to compute the exponent. Also, we need the rest mass of the electron $m = 511\,\text{keV}/c^2$ and Planck's constant $\hbar = 0.1973\text{keV}\cdot\text{nm}/c$. It is typical for this type of estimate to deal with very small quantities that are often not suitable for handheld calculators. To make correct estimates of orders, we make the conversion $e^y = 10^{y/\ln 10}$.

Solution

Constants:

$$\frac{2m}{\hbar^2} = \frac{2(511\,\text{keV}/c^2)}{(0.1973\text{keV}\cdot\text{nm}/c)^2} = 26,\,254\frac{1}{\text{keV}\cdot(\text{nm})^2},$$

$$\beta = \sqrt{\frac{2m}{\hbar^2}(U_0 - E)} = \sqrt{26,\,254\frac{(10.0\,\text{eV} - E)}{\text{keV}\cdot(\text{nm})^2}} = \sqrt{26.254(10.0\,\text{eV} - E)/\text{eV}}\frac{1}{\text{nm}}.$$

For a lower-energy electron with $E_1 = 7.00\,\text{eV}$:

$$\beta_1 = \sqrt{26.254(10.00\,\text{eV} - E_1)/\text{eV}}\frac{1}{\text{nm}} = \sqrt{26.254(10.00 - 7.00)}\frac{1}{\text{nm}} = \frac{8.875}{\text{nm}},$$

$$T(L,\,E_1) = 16\frac{E_1}{U_0}\left(1 - \frac{E_1}{U_0}\right)e^{-2\beta_1 L} = 16\frac{7}{10}\left(1 - \frac{7}{10}\right)e^{-17.75\,L/\text{nm}} = 3.36e^{-17.75\,L/\text{nm}}.$$

For a higher-energy electron with $E_2 = 9.00\,\text{eV}$:

$$\beta_2 = \sqrt{26.254(10.00\,\text{eV} - E_2)/\text{eV}}\frac{1}{\text{nm}} = \sqrt{26.254(10.00 - 9.00)}\frac{1}{\text{nm}} = \frac{5.124}{\text{nm}},$$

$$T(L,\,E_2) = 16\frac{E_2}{U_0}\left(1 - \frac{E_2}{U_0}\right)e^{-2\beta_2 L} = 16\frac{9}{10}\left(1 - \frac{9}{10}\right)e^{-5.12\,L/\text{nm}} = 1.44e^{-5.12\,L/\text{nm}}.$$

For a broad barrier with $L_1 = 5.00\,\text{nm}$:

$$T(L_1,\,E_1) = 3.36e^{-17.75\,L_1/\text{nm}} = 3.36e^{-17.75\cdot 5.00\,\text{nm/nm}} = 3.36e^{-88} = 3.36(6.2\times 10^{-39}) = 2.1\%\times 10^{-36},$$

$$T(L_1,\,E_2) = 1.44e^{-5.12\,L_1/\text{nm}} = 1.44e^{-5.12\cdot 5.00\,\text{nm/nm}} = 1.44e^{-25.6} = 1.44(7.62\times 10^{-12}) = 1.1\%\times 10^{-9}.$$

For a narrower barrier with $L_2 = 1.00\,\text{nm}$:

$$T(L_2,\,E_1) = 3.36e^{-17.75\,L_2/\text{nm}} = 3.36e^{-17.75\cdot 1.00\,\text{nm/nm}} = 3.36e^{-17.75} = 3.36(5.1\times 10^{-7}) = 1.7\%\times 10^{-4},$$

$$T(L_2,\,E_2) = 1.44e^{-5.12\,L_2/\text{nm}} = 1.44e^{-5.12\cdot 1.00\,\text{nm/nm}} = 1.44e^{-5.12} = 1.44(5.98\times 10^{-3}) = 0.86\%.$$

Significance

We see from these estimates that the probability of tunneling is affected more by the width of the potential barrier than by the energy of an incident particle. In today's technologies, we can manipulate individual atoms on metal surfaces to create potential barriers that are fractions of a nanometer, giving rise to measurable tunneling currents. One of many applications of this technology is the scanning tunneling microscope (STM), which we discuss later in this section.

7.10 Check Your Understanding A proton with kinetic energy 1.00 eV is incident on a square potential barrier with height 10.00 eV. If the proton is to have the same transmission probability as an electron of the same energy, what must the width of the barrier be relative to the barrier width encountered by an electron?

Radioactive Decay

In 1928, Gamow identified quantum tunneling as the mechanism responsible for the radioactive decay of atomic nuclei. He observed that some isotopes of thorium, uranium, and bismuth disintegrate by emitting α-particles (which are doubly ionized helium atoms or, simply speaking, helium nuclei). In the process of emitting an α-particle, the original nucleus is transformed into a new nucleus that has two fewer neutrons and two fewer protons than the original nucleus. The α-particles emitted by one isotope have approximately the same kinetic energies. When we look at variations of these energies among isotopes of various elements, the lowest kinetic energy is about 4 MeV and the highest is about 9 MeV, so these energies are of the same order of magnitude. This is about where the similarities between various isotopes end.

When we inspect half-lives (a half-life is the time in which a radioactive sample loses half of its nuclei due to decay), different isotopes differ widely. For example, the half-life of polonium-214 is $160\,\mu\text{s}$ and the half-life of uranium is 4.5 billion years. Gamow explained this variation by considering a 'spherical-box' model of the nucleus, where α-particles can bounce back and forth between the walls as free particles. The confinement is provided by a strong nuclear potential at a spherical wall of the box. The thickness of this wall, however, is not infinite but finite, so in principle, a nuclear particle has a chance to escape this nuclear confinement. On the inside wall of the confining barrier is a high nuclear potential that keeps the α-particle in a small confinement. But when an α-particle gets out to the other side of this wall, it is subject to electrostatic Coulomb repulsion and moves away from the nucleus. This idea is illustrated in **Figure 7.18**. The width L of the potential barrier that separates an α-particle from the outside world depends on the particle's kinetic energy E. This width is the distance between the point marked by the nuclear radius R and the point R_0 where an α-particle emerges on the other side of the barrier, $L = R_0 - R$. At the distance R_0, its kinetic energy must at least match the electrostatic energy of repulsion, $E = (4\pi\varepsilon_0)^{-1} Ze^2/R_0$ (where $+Ze$ is the charge of the nucleus). In this way we can estimate the width of the nuclear barrier,

$$L = \frac{e^2}{4\pi\varepsilon_0}\frac{Z}{E} - R.$$

We see from this estimate that the higher the energy of α-particle, the narrower the width of the barrier that it is to tunnel through. We also know that the width of the potential barrier is the most important parameter in tunneling probability. Thus, highly energetic α-particles have a good chance to escape the nucleus, and, for such nuclei, the nuclear disintegration half-life is short. Notice that this process is highly nonlinear, meaning a small increase in the α-particle energy has a disproportionately large enhancing effect on the tunneling probability and, consequently, on shortening the half-life. This explains why the half-life of polonium that emits 8-MeV α-particles is only hundreds of milliseconds and the half-life of uranium that emits 4-MeV α-particles is billions of years.

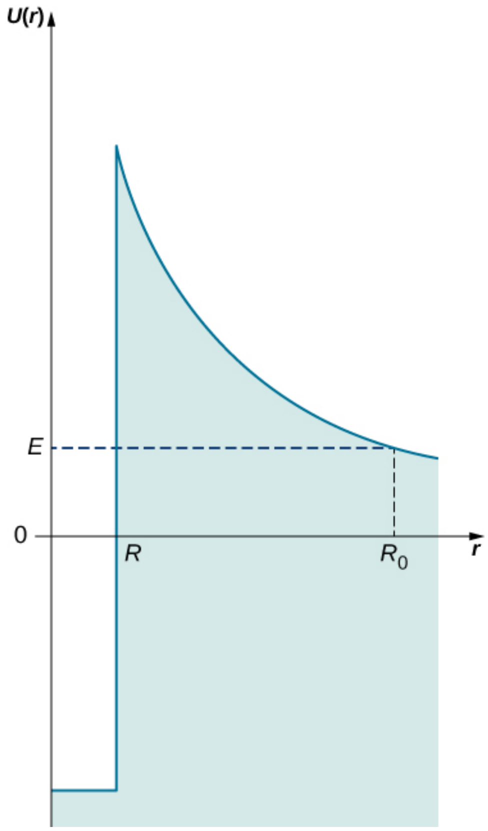


Figure 7.18 The potential energy barrier for an α-particle bound in the nucleus: To escape from the nucleus, an α-particle with energy *E* must tunnel across the barrier from distance *R* to distance R_0 away from the center.

Field Emission

Field emission is a process of emitting electrons from conducting surfaces due to a strong external electric field that is applied in the direction normal to the surface (**Figure 7.19**). As we know from our study of electric fields in earlier chapters, an applied external electric field causes the electrons in a conductor to move to its surface and stay there as long as the present external field is not excessively strong. In this situation, we have a constant electric potential throughout the inside of the conductor, including its surface. In the language of potential energy, we say that an electron inside the conductor has a constant potential energy $U(x) = -U_0$ (here, the *x* means inside the conductor). In the situation represented in **Figure 7.19**, where the external electric field is uniform and has magnitude E_g, if an electron happens to be outside the conductor at a distance *x* away from its surface, its potential energy would have to be $U(x) = -eE_g x$ (here, *x* denotes distance to the surface). Taking the origin at the surface, so that $x = 0$ is the location of the surface, we can represent the potential energy of conduction electrons in a metal as the potential energy barrier shown in **Figure 7.20**. In the absence of the external field, the potential energy becomes a step barrier defined by $U(x \le 0) = -U_0$ and by $U(x > 0) = 0$.

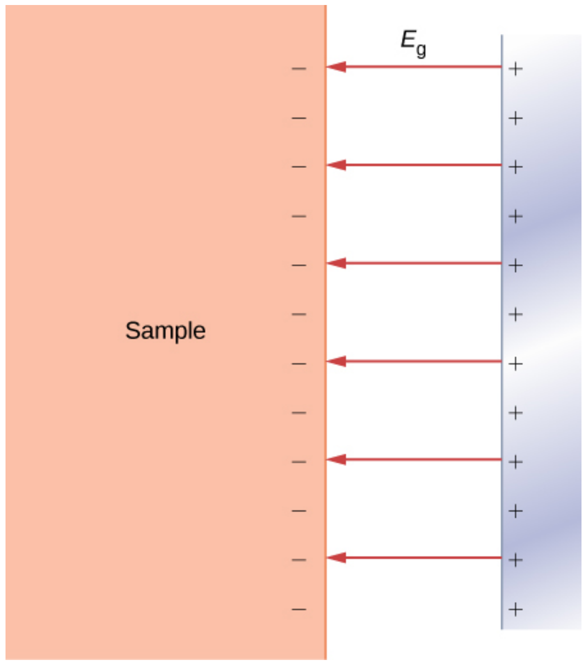


Figure 7.19 A normal-direction external electric field at the surface of a conductor: In a strong field, the electrons on a conducting surface may get detached from it and accelerate against the external electric field away from the surface.

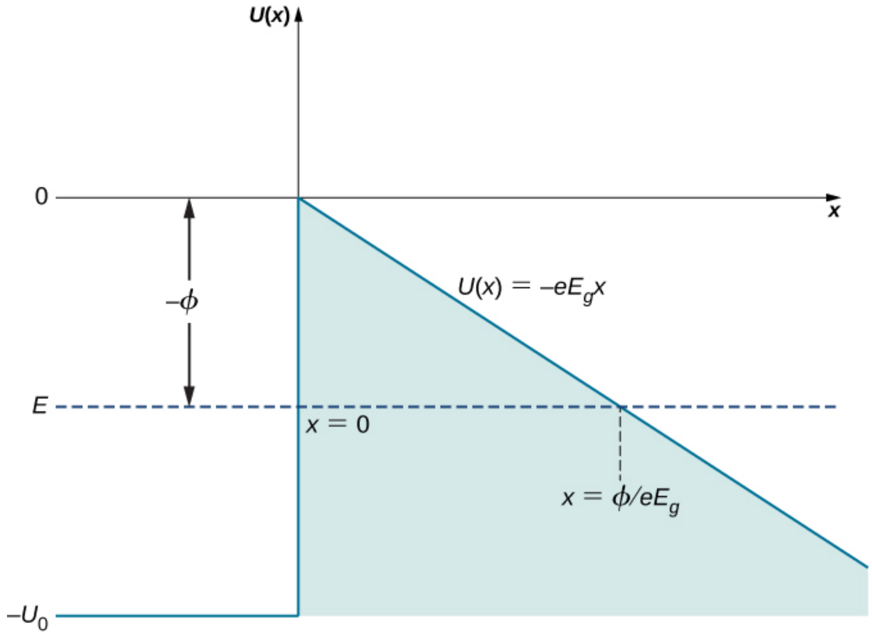


Figure 7.20 The potential energy barrier at the surface of a metallic conductor in the presence of an external uniform electric field E_g normal to the surface: It becomes a step-function barrier when the external field is removed. The work function of the metal is indicated by ϕ.

When an external electric field is strong, conduction electrons at the surface may get detached from it and accelerate along electric field lines in a direction antiparallel to the external field, away from the surface. In short, conduction electrons may escape from the surface. The field emission can be understood as the quantum tunneling of conduction electrons through the potential barrier at the conductor's surface. The physical principle at work here is very similar to the mechanism of α-emission from a radioactive nucleus.

Suppose a conduction electron has a kinetic energy E (the average kinetic energy of an electron in a metal is the work function ϕ for the metal and can be measured, as discussed for the photoelectric effect in **Photons and Matter Waves**), and an external electric field can be locally approximated by a uniform electric field of strength E_g. The width L of the potential barrier that the electron must cross is the distance from the conductor's surface to the point outside the surface where its kinetic energy matches the value of its potential energy in the external field. In **Figure 7.20**, this distance is measured along the dashed horizontal line $U(x) = E$ from $x = 0$ to the intercept with $U(x) = -eE_g x$, so the barrier width is

$$L = \frac{e^{-1}E}{E_g} = \frac{e^{-1}\phi}{E_g}.$$

We see that L is inversely proportional to the strength E_g of an external field. When we increase the strength of the external field, the potential barrier outside the conductor becomes steeper and its width decreases for an electron with a given kinetic energy. In turn, the probability that an electron will tunnel across the barrier (conductor surface) becomes exponentially larger. The electrons that emerge on the other side of this barrier form a current (tunneling-electron current) that can be detected above the surface. The tunneling-electron current is proportional to the tunneling probability. The tunneling probability depends nonlinearly on the barrier width L, and L can be changed by adjusting E_g. Therefore, the tunneling-electron current can be tuned by adjusting the strength of an external electric field at the surface. When the strength of an external electric field is constant, the tunneling-electron current has different values at different elevations L above the surface.

The quantum tunneling phenomenon at metallic surfaces, which we have just described, is the physical principle behind the operation of the **scanning tunneling microscope (STM)**, invented in 1981 by Gerd Binnig and Heinrich Rohrer. The STM device consists of a scanning tip (a needle, usually made of tungsten, platinum-iridium, or gold); a piezoelectric device that controls the tip's elevation in a typical range of 0.4 to 0.7 nm above the surface to be scanned; some device that controls the motion of the tip along the surface; and a computer to display images. While the sample is kept at a suitable voltage bias, the scanning tip moves along the surface (**Figure 7.21**), and the tunneling-electron current between the tip and the surface is registered at each position. The amount of the current depends on the probability of electron tunneling from the surface to the tip, which, in turn, depends on the elevation of the tip above the surface. Hence, at each tip position, the distance from the tip to the surface is measured by measuring how many electrons tunnel out from the surface to the tip. This method can give an unprecedented resolution of about 0.001 nm, which is about 1% of the average diameter of an atom. In this way, we can see individual atoms on the surface, as in the image of a carbon nanotube in **Figure 7.22**.

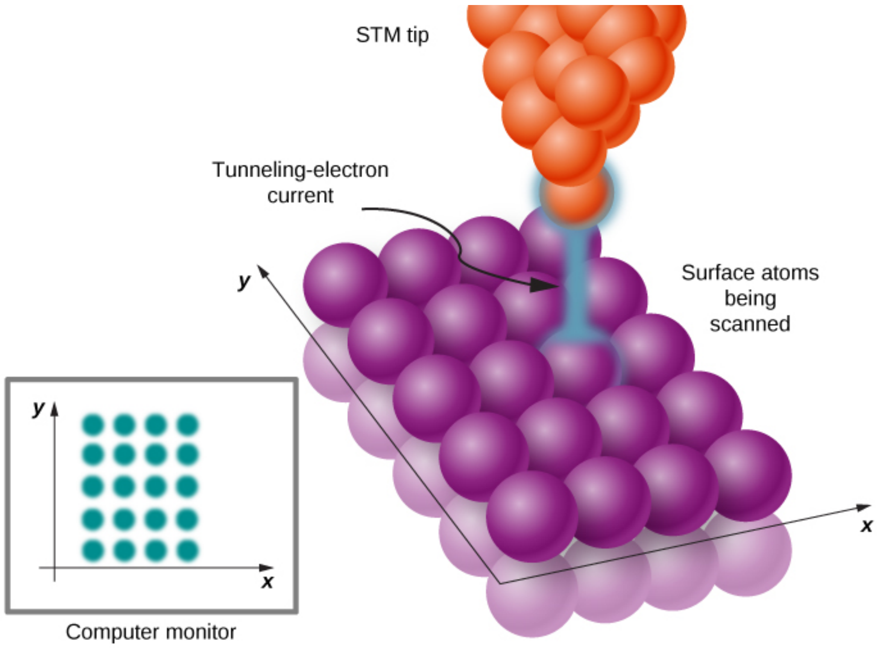


Figure 7.21 In STM, a surface at a constant potential is being scanned by a narrow tip moving along the surface. When the STM tip moves close to surface atoms, electrons can tunnel from the surface to the tip. This tunneling-electron current is continually monitored while the tip is in motion. The amount of current at location (x,y) gives information about the elevation of the tip above the surface at this location. In this way, a detailed topographical map of the surface is created and displayed on a computer monitor.

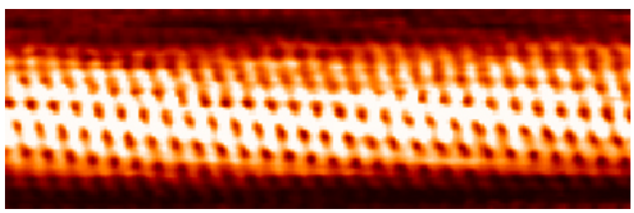


Figure 7.22 An STM image of a carbon nanotube: Atomic-scale resolution allows us to see individual atoms on the surface. STM images are in gray scale, and coloring is added to bring up details to the human eye.

Resonant Quantum Tunneling

Quantum tunneling has numerous applications in semiconductor devices such as electronic circuit components or integrated circuits that are designed at nanoscales; hence, the term ‘ **nanotechnology**.’ For example, a diode (an electric-circuit element that causes an electron current in one direction to be different from the current in the opposite direction, when the polarity of the bias voltage is reversed) can be realized by a tunneling junction between two different types of semiconducting materials. In such a **tunnel diode**, electrons tunnel through a single potential barrier at a contact between two different semiconductors. At the junction, tunneling-electron current changes nonlinearly with the applied potential difference across the junction and may rapidly decrease as the bias voltage is increased. This is unlike the Ohm’s law behavior that we are familiar with in household circuits. This kind of rapid behavior (caused by quantum tunneling) is desirable in high-speed electronic devices.

Another kind of electronic nano-device utilizes **resonant tunneling** of electrons through potential barriers that occur in quantum dots. A **quantum dot** is a small region of a semiconductor nanocrystal that is grown, for example, in a silicon or aluminum arsenide crystal. **Figure 7.23**(a) shows a quantum dot of gallium arsenide embedded in an aluminum arsenide wafer. The quantum-dot region acts as a potential well of a finite height (shown in **Figure 7.23**(b)) that has two finite-height potential barriers at dot boundaries. Similarly, as for a quantum particle in a box (that is, an infinite potential well), lower-lying energies of a quantum particle trapped in a finite-height potential well are quantized. The difference between the box and the well potentials is that a quantum particle in a box has an infinite number of quantized energies and is trapped in the box indefinitely, whereas a quantum particle trapped in a potential well has a finite number of quantized energy levels and can tunnel through potential barriers at well boundaries to the outside of the well. Thus, a quantum dot of gallium arsenide sitting in aluminum arsenide is a potential well where low-lying energies of an electron are quantized, indicated as E_{dot} in part (b) in the figure. When the energy E_{electron} of an electron in the outside region of the dot does not match its energy E_{dot} that it would have in the dot, the electron does not tunnel through the region of the dot and there is no current through such a circuit element, even if it were kept at an electric voltage difference (bias). However, when this voltage bias is changed in such a way that one of the barriers is lowered, so that E_{dot} and E_{electron} become aligned, as seen in part (c) of the figure, an electron current flows through the dot. When the voltage bias is now increased, this alignment is lost and the current stops flowing. When the voltage bias is increased further, the electron tunneling becomes improbable until the bias voltage reaches a value for which the outside electron energy matches the next electron energy level in the dot. The word 'resonance' in the device name means that the tunneling-electron current occurs only when a selected energy level is matched by tuning an applied voltage bias, such as in the operation mechanism of the **resonant-tunneling diode** just described. Resonant-tunneling diodes are used as super-fast nano-switches.

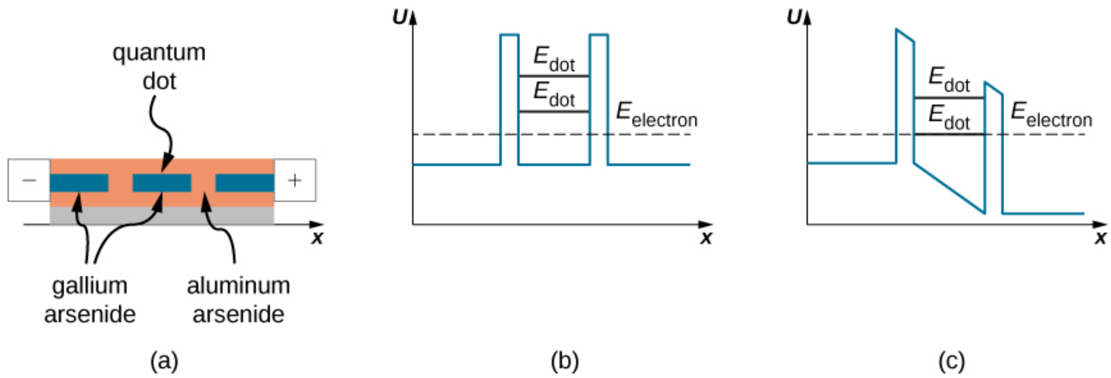


Figure 7.23 Resonant-tunneling diode: (a) A quantum dot of gallium arsenide embedded in aluminum arsenide. (b) Potential well consisting of two potential barriers of a quantum dot with no voltage bias. Electron energies E_{electron} in aluminum arsenide are not aligned with their energy levels E_{dot} in the quantum dot, so electrons do not tunnel through the dot. (c) Potential well of the dot with a voltage bias across the device. A suitably tuned voltage difference distorts the well so that electron-energy levels in the dot are aligned with their energies in aluminum arsenide, causing the electrons to tunnel through the dot.

CHAPTER 7 REVIEW

KEY TERMS

anti-symmetric function odd function

Born interpretation states that the square of a wave function is the probability density

complex function function containing both real and imaginary parts

Copenhagen interpretation states that when an observer *is not* looking or when a measurement is not being made, the particle has many values of measurable quantities, such as position

correspondence principle in the limit of large energies, the predictions of quantum mechanics agree with the predictions of classical mechanics

energy levels states of definite energy, often represented by horizontal lines in an energy “ladder” diagram

energy quantum number index that labels the allowed energy states

energy-time uncertainty principle energy-time relation for uncertainties in the simultaneous measurements of the energy of a quantum state and of its lifetime

even function in one dimension, a function symmetric with the origin of the coordinate system

expectation value average value of the physical quantity assuming a large number of particles with the same wave function

field emission electron emission from conductor surfaces when a strong external electric field is applied in normal direction to conductor’s surface

ground state energy lowest energy state in the energy spectrum

Heisenberg’s uncertainty principle places limits on what can be known from a simultaneous measurements of position and momentum; states that if the uncertainty on position is small then the uncertainty on momentum is large, and vice versa

infinite square well potential function that is zero in a fixed range and infinitely beyond this range

momentum operator operator that corresponds to the momentum of a particle

nanotechnology technology that is based on manipulation of nanostructures such as molecules or individual atoms to produce nano-devices such as integrated circuits

normalization condition requires that the probability density integrated over the entire physical space results in the number one

odd function in one dimension, a function antisymmetric with the origin of the coordinate system

position operator operator that corresponds to the position of a particle

potential barrier potential function that rises and falls with increasing values of position

principal quantum number energy quantum number

probability density square of the particle’s wave function

quantum dot small region of a semiconductor nanocrystal embedded in another semiconductor nanocrystal, acting as a potential well for electrons

quantum tunneling phenomenon where particles penetrate through a potential energy barrier with a height greater than the total energy of the particles

resonant tunneling tunneling of electrons through a finite-height potential well that occurs only when electron energies match an energy level in the well, occurs in quantum dots

resonant-tunneling diode quantum dot with an applied voltage bias across it

scanning tunneling microscope (STM) device that utilizes quantum-tunneling phenomenon at metallic surfaces to obtain images of nanoscale structures

Schrödinger's time-dependent equation equation in space and time that allows us to determine wave functions of a quantum particle

Schrödinger's time-independent equation equation in space that allows us to determine wave functions of a quantum particle; this wave function must be multiplied by a time-modulation factor to obtain the time-dependent wave function

standing wave state stationary state for which the real and imaginary parts of $\Psi(x, t)$ oscillate up and down like a standing wave (often modeled with sine and cosine functions)

state reduction hypothetical process in which an observed or detected particle "jumps into" a definite state, often described in terms of the collapse of the particle's wave function

stationary state state for which the probability density function, $|\Psi(x, t)|^2$, does not vary in time

time-modulation factor factor $e^{-i\omega t}$ that multiplies the time-independent wave function when the potential energy of the particle is time independent

transmission probability also called tunneling probability, the probability that a particle will tunnel through a potential barrier

tunnel diode electron tunneling-junction between two different semiconductors

tunneling probability also called transmission probability, the probability that a particle will tunnel through a potential barrier

wave function function that represents the quantum state of a particle (quantum system)

wave function collapse equivalent to state reduction

wave packet superposition of many plane matter waves that can be used to represent a localized particle

KEY EQUATIONS

Normalization condition in one dimension	$P(x = -\infty, +\infty) = \int_{-\infty}^{\infty} \|\Psi(x, t)\|^2 dx = 1$
Probability of finding a particle in a narrow interval of position in one dimension $(x, x + dx)$	$P(x, x + dx) = \Psi^*(x, t)\Psi(x, t)dx$
Expectation value of position in one dimension	$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^*(x, t)x\Psi(x, t)dx$
Heisenberg's position-momentum uncertainty principle	$\Delta x \Delta p \geq \frac{\hbar}{2}$
Heisenberg's energy-time uncertainty principle	$\Delta E \Delta t \geq \frac{\hbar}{2}$
Schrödinger's time-dependent equation	$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x, t)}{\partial x^2} + U(x, t)\Psi(x, t) = i\hbar\frac{\partial^2 \Psi(x, t)}{\partial t}$
General form of the wave function for a time-independent potential in one dimension	$\Psi(x, t) = \psi(x)e^{-i\omega t}$
Schrödinger's time-independent equation	$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$
Schrödinger's equation (free particle)	$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} = E\psi(x)$
Allowed energies (particle in box of length L)	$E_n = n^2\frac{\pi^2\hbar^2}{2mL^2}, n = 1, 2, 3, \ldots$

Stationary states (particle in a box of length *L*)	$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L},\ n = 1,\ 2,\ 3,\ ...$
Potential-energy function of a harmonic oscillator	$U(x) = \frac{1}{2} m\omega^2 x^2$
Stationary Schrödinger equation	$-\frac{\hbar}{2m} \frac{d^2\psi(x)}{dx^2} + \frac{1}{2} m\omega^2 x^2 \psi(x) = E\psi(x)$
The energy spectrum	$E_n = \left(n + \frac{1}{2}\right)\hbar\omega,\ n = 0,\ 1,\ 2,\ 3,\ ...$
The energy wave functions	$\psi_n(x) = N_n e^{-\beta^2 x^2/2} H_n(\beta x),\ n = 0,\ 1,\ 2,\ 3,\ ...$
Potential barrier	$U(x) = \begin{cases} 0, & \text{when } x < 0 \\ U_0, & \text{when } 0 \le x \le L \\ 0, & \text{when } x > L \end{cases}$
Definition of the transmission coefficient	$T(L,\ E) = \frac{\lvert\psi_{\text{tra}}(x)\rvert^2}{\lvert\psi_{\text{in}}(x)\rvert^2}$
A parameter in the transmission coefficient	$\beta^2 = \frac{2m}{\hbar^2}(U_0 - E)$
Transmission coefficient, exact	$T(L,\ E) = \frac{1}{\cosh^2 \beta L + (\gamma/2)^2 \sinh^2 \beta L}$
Transmission coefficient, approximate	$T(L,\ E) = 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right)e^{-2\beta L}$

SUMMARY

7.1 Wave Functions

- In quantum mechanics, the state of a physical system is represented by a wave function.
- In Born's interpretation, the square of the particle's wave function represents the probability density of finding the particle around a specific location in space.
- Wave functions must first be normalized before using them to make predictions.
- The expectation value is the average value of a quantity that requires a wave function and an integration.

7.2 The Heisenberg Uncertainty Principle

- The Heisenberg uncertainty principle states that it is impossible to simultaneously measure the *x*-components of position and of momentum of a particle with an arbitrarily high precision. The product of experimental uncertainties is always larger than or equal to $\hbar/2$.
- The limitations of this principle have nothing to do with the quality of the experimental apparatus but originate in the wave-like nature of matter.
- The energy-time uncertainty principle expresses the experimental observation that a quantum state that exists only for a short time cannot have a definite energy.

7.3 The Schrödinger Equation

- The Schrödinger equation is the fundamental equation of wave quantum mechanics. It allows us to make predictions about wave functions.

- When a particle moves in a time-independent potential, a solution of the time-dependent Schrödinger equation is a product of a time-independent wave function and a time-modulation factor.
- The Schrödinger equation can be applied to many physical situations.

7.4 The Quantum Particle in a Box

- Energy states of a quantum particle in a box are found by solving the time-independent Schrödinger equation.
- To solve the time-independent Schrödinger equation for a particle in a box and find the stationary states and allowed energies, we require that the wave function terminate at the box wall.
- Energy states of a particle in a box are quantized and indexed by principal quantum number.
- The quantum picture differs significantly from the classical picture when a particle is in a low-energy state of a low quantum number.
- In the limit of high quantum numbers, when the quantum particle is in a highly excited state, the quantum description of a particle in a box coincides with the classical description, in the spirit of Bohr's correspondence principle.

7.5 The Quantum Harmonic Oscillator

- The quantum harmonic oscillator is a model built in analogy with the model of a classical harmonic oscillator. It models the behavior of many physical systems, such as molecular vibrations or wave packets in quantum optics.
- The allowed energies of a quantum oscillator are discrete and evenly spaced. The energy spacing is equal to Planck's energy quantum.
- The ground state energy is larger than zero. This means that, unlike a classical oscillator, a quantum oscillator is never at rest, even at the bottom of a potential well, and undergoes quantum fluctuations.
- The stationary states (states of definite energy) have nonzero values also in regions beyond classical turning points. When in the ground state, a quantum oscillator is most likely to be found around the position of the minimum of the potential well, which is the least-likely position for a classical oscillator.
- For high quantum numbers, the motion of a quantum oscillator becomes more similar to the motion of a classical oscillator, in accordance with Bohr's correspondence principle.

7.6 The Quantum Tunneling of Particles through Potential Barriers

- A quantum particle that is incident on a potential barrier of a finite width and height may cross the barrier and appear on its other side. This phenomenon is called 'quantum tunneling.' It does not have a classical analog.
- To find the probability of quantum tunneling, we assume the energy of an incident particle and solve the stationary Schrödinger equation to find wave functions inside and outside the barrier. The tunneling probability is a ratio of squared amplitudes of the wave past the barrier to the incident wave.
- The tunneling probability depends on the energy of the incident particle relative to the height of the barrier and on the width of the barrier. It is strongly affected by the width of the barrier in a nonlinear, exponential way so that a small change in the barrier width causes a disproportionately large change in the transmission probability.
- Quantum-tunneling phenomena govern radioactive nuclear decays. They are utilized in many modern technologies such as STM and nano-electronics. STM allows us to see individual atoms on metal surfaces. Electron-tunneling devices have revolutionized electronics and allow us to build fast electronic devices of miniature sizes.

CONCEPTUAL QUESTIONS

7.1 Wave Functions

1. What is the physical unit of a wave function, $\Psi(x, t)$? What is the physical unit of the square of this wave function?

2. Can the magnitude of a wave function $(\Psi^*(x, t)\Psi(x, t))$ be a negative number? Explain.

3. What kind of physical quantity does a wave function of an electron represent?

4. What is the physical meaning of a wave function of a particle?

5. What is the meaning of the expression "expectation value?" Explain.

7.2 The Heisenberg Uncertainty Principle

6. If the formalism of quantum mechanics is 'more exact' than that of classical mechanics, why don't we use quantum mechanics to describe the motion of a leaping frog? Explain.

7. Can the de Broglie wavelength of a particle be known precisely? Can the position of a particle be known precisely?

8. Can we measure the energy of a free localized particle with complete precision?

9. Can we measure both the position and momentum of a particle with complete precision?

7.3 The Schrödinger Equation

10. What is the difference between a wave function $\psi(x,\ y,\ z)$ and a wave function $\Psi(x,\ y,\ z,\ t)$ for the same particle?

11. If a quantum particle is in a stationary state, does it mean that it does not move?

12. Explain the difference between time-dependent and -independent Schrödinger's equations.

13. Suppose a wave function is discontinuous at some point. Can this function represent a quantum state of some physical particle? Why? Why not?

7.4 The Quantum Particle in a Box

14. Using the quantum particle in a box model, describe how the possible energies of the particle are related to the size of the box.

15. Is it possible that when we measure the energy of a quantum particle in a box, the measurement may return a smaller value than the ground state energy? What is the highest value of the energy that we can measure for this particle?

16. For a quantum particle in a box, the first excited state (Ψ_2) has zero value at the midpoint position in the box, so that the probability density of finding a particle at this point is exactly zero. Explain what is wrong with the following reasoning: "If the probability of finding a quantum particle at the midpoint is zero, the particle is never at this point, right? How does it come then that the particle can cross this point on its way from the left side to the right side of the box?

7.5 The Quantum Harmonic Oscillator

17. Is it possible to measure energy of $0.75\hbar\omega$ for a quantum harmonic oscillator? Why? Why not? Explain.

18. Explain the connection between Planck's hypothesis of energy quanta and the energies of the quantum harmonic oscillator.

19. If a classical harmonic oscillator can be at rest, why can the quantum harmonic oscillator never be at rest? Does this violate Bohr's correspondence principle?

20. Use an example of a quantum particle in a box or a quantum oscillator to explain the physical meaning of Bohr's correspondence principle.

21. Can we simultaneously measure position and energy of a quantum oscillator? Why? Why not?

7.6 The Quantum Tunneling of Particles through Potential Barriers

22. When an electron and a proton of the same kinetic energy encounter a potential barrier of the same height and width, which one of them will tunnel through the barrier more easily? Why?

23. What decreases the tunneling probability most: doubling the barrier width or halving the kinetic energy of the incident particle?

24. Explain the difference between a box-potential and a potential of a quantum dot.

25. Can a quantum particle 'escape' from an infinite potential well like that in a box? Why? Why not?

26. A tunnel diode and a resonant-tunneling diode both utilize the same physics principle of quantum tunneling. In what important way are they different?

PROBLEMS

7.1 Wave Functions

27. Compute $|\Psi(x,\ t)|^2$ for the function $\Psi(x,\ t) = \psi(x) \sin \omega t$, where ω is a real constant.

28. Given the complex-valued function $f(x,\ y) = (x - iy)/(x + iy)$, calculate $|f(x,\ y)|^2$.

29. Which one of the following functions, and why, qualifies to be a wave function of a particle that can move along the entire real axis? (a) $\psi(x) = Ae^{-x^2}$; (b) $\psi(x) = Ae^{-x}$; (c) $\psi(x) = A \tan x$; (d) $\psi(x) = A(\sin x)/x$; (e) $\psi(x) = Ae^{-|x|}$.

30. A particle with mass *m* moving along the *x*-axis and its quantum state is represented by the following wave function:

$$\Psi(x,\ t) = \begin{cases} 0, & x < 0, \\ Axe^{-\alpha x} e^{-iEt/\hbar}, & x \geq 0, \end{cases}$$

where $\alpha = 2.0 \times 10^{10}\,\text{m}^{-1}$. (a) Find the normalization constant. (b) Find the probability that the particle can be found on the interval $0 \leq x \leq L$. (c) Find the expectation value of position. (d) Find the expectation value of kinetic energy.

31. A wave function of a particle with mass *m* is given by

$$\psi(x) = \begin{cases} A \cos \alpha x, & -\frac{\pi}{2\alpha} \leq x \leq +\frac{\pi}{2\alpha}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\alpha = 1.00 \times 10^{10}/\text{m}$. (a) Find the normalization constant. (b) Find the probability that the particle can be found on the interval $0 \leq x \leq 0.5 \times 10^{-10}\,\text{m}$. (c) Find the particle's average position. (d) Find its average momentum. (e) Find its average kinetic energy $-0.5 \times 10^{-10}\,\text{m} \leq x \leq +0.5 \times 10^{-10}\,\text{m}$.

7.2 The Heisenberg Uncertainty Principle

32. A velocity measurement of an α-particle has been performed with a precision of 0.02 mm/s. What is the minimum uncertainty in its position?

33. A gas of helium atoms at 273 K is in a cubical container with 25.0 cm on a side. (a) What is the minimum uncertainty in momentum components of helium atoms? (b) What is the minimum uncertainty in velocity components? (c) Find the ratio of the uncertainties in (b) to the mean speed of an atom in each direction.

34. If the uncertainty in the y-component of a proton's position is 2.0 pm, find the minimum uncertainty in the simultaneous measurement of the proton's y-component of velocity. What is the minimum uncertainty in the simultaneous measurement of the proton's x-component of velocity?

35. Some unstable elementary particle has a rest energy of 80.41 GeV and an uncertainty in rest energy of 2.06 GeV. Estimate the lifetime of this particle.

36. An atom in a metastable state has a lifetime of 5.2 ms. Find the minimum uncertainty in the measurement of energy of the excited state.

37. Measurements indicate that an atom remains in an excited state for an average time of 50.0 ns before making a transition to the ground state with the simultaneous emission of a 2.1-eV photon. (a) Estimate the uncertainty in the frequency of the photon. (b) What fraction of the photon's average frequency is this?

38. Suppose an electron is confined to a region of length 0.1 nm (of the order of the size of a hydrogen atom) and its kinetic energy is equal to the ground state energy of the hydrogen atom in Bohr's model (13.6 eV). (a) What is the minimum uncertainty of its momentum? What fraction of its momentum is it? (b) What would the uncertainty in kinetic energy of this electron be if its momentum were equal to your answer in part (a)? What fraction of its kinetic energy is it?

7.3 The Schrӧdinger Equation

39. Combine **Equation 7.17** and **Equation 7.18** to show $k^2 = \frac{\omega^2}{c^2}$.

40. Show that $\Psi(x,\ t) = Ae^{i(kx - \omega t)}$ is a valid solution to Schrӧdinger's time-dependent equation.

41. Show that $\Psi(x,\ t) = A \sin(kx - \omega t)$ and $\Psi(x,\ t) = A \cos(kx - \omega t)$ do not obey Schrӧdinger's time-dependent equation.

42. Show that when $\Psi_1(x, t)$ and $\Psi_2(x, t)$ are solutions to the time-dependent Schrödinger equation and *A*,*B* are numbers, then a function $\Psi(x, t)$ that is a superposition of these functions is also a solution: $\Psi(x, t) = A\Psi_1(x, t) + B\Psi_1(x, t)$.

43. A particle with mass *m* is described by the following wave function: $\psi(x) = A\cos kx + B\sin kx$, where *A*, *B*, and *k* are constants. Assuming that the particle is free, show that this function is the solution of the stationary Schrödinger equation for this particle and find the energy that the particle has in this state.

44. Find the expectation value of the kinetic energy for the particle in the state, $\Psi(x, t) = Ae^{i(kx - \omega t)}$. What conclusion can you draw from your solution?

45. Find the expectation value of the square of the momentum squared for the particle in the state, $\Psi(x, t) = Ae^{i(kx - \omega t)}$. What conclusion can you draw from your solution?

46. A free proton has a wave function given by $\Psi(x, t) = Ae^{i(5.02 \times 10^{11} x - 8.00 \times 10^{15} t)}$.

The coefficient of *x* is inverse meters (m^{-1}) and the coefficient on *t* is inverse seconds (s^{-1}). Find its momentum and energy.

7.4 The Quantum Particle in a Box

47. Assume that an electron in an atom can be treated as if it were confined to a box of width $2.0\ \text{Å}$. What is the ground state energy of the electron? Compare your result to the ground state kinetic energy of the hydrogen atom in the Bohr's model of the hydrogen atom.

48. Assume that a proton in a nucleus can be treated as if it were confined to a one-dimensional box of width 10.0 fm. (a) What are the energies of the proton when it is in the states corresponding to $n = 1$, $n = 2$, and $n = 3$? (b) What are the energies of the photons emitted when the proton makes the transitions from the first and second excited states to the ground state?

49. An electron confined to a box has the ground state energy of 2.5 eV. What is the width of the box?

50. What is the ground state energy (in eV) of a proton confined to a one-dimensional box the size of the uranium nucleus that has a radius of approximately 15.0 fm?

51. What is the ground state energy (in eV) of an α-particle confined to a one-dimensional box the size of the uranium nucleus that has a radius of approximately 15.0 fm?

52. To excite an electron in a one-dimensional box from its first excited state to its third excited state requires 20.0 eV. What is the width of the box?

53. An electron confined to a box of width 0.15 nm by infinite potential energy barriers emits a photon when it makes a transition from the first excited state to the ground state. Find the wavelength of the emitted photon.

54. If the energy of the first excited state of the electron in the box is 25.0 eV, what is the width of the box?

55. Suppose an electron confined to a box emits photons. The longest wavelength that is registered is 500.0 nm. What is the width of the box?

56. Hydrogen H_2 molecules are kept at 300.0 K in a cubical container with a side length of 20.0 cm. Assume that you can treat the molecules as though they were moving in a one-dimensional box. (a) Find the ground state energy of the hydrogen molecule in the container. (b) Assume that the molecule has a thermal energy given by $k_B T/2$ and find the corresponding quantum number *n* of the quantum state that would correspond to this thermal energy.

57. An electron is confined to a box of width 0.25 nm. (a) Draw an energy-level diagram representing the first five states of the electron. (b) Calculate the wavelengths of the emitted photons when the electron makes transitions between the fourth and the second excited states, between the second excited state and the ground state, and between the third and the second excited states.

58. An electron in a box is in the ground state with energy 2.0 eV. (a) Find the width of the box. (b) How much energy is needed to excite the electron to its first excited state? (c) If the electron makes a transition from an excited state to the ground state with the simultaneous emission of 30.0-eV photon, find the quantum number of the excited state?

7.5 The Quantum Harmonic Oscillator

59. Show that the two lowest energy states of the simple harmonic oscillator, $\psi_0(x)$ and $\psi_1(x)$ from **Equation 7.57**, satisfy **Equation 7.55**.

60. If the ground state energy of a simple harmonic oscillator is 1.25 eV, what is the frequency of its motion?

61. When a quantum harmonic oscillator makes a transition from the $(n+1)$ state to the n state and emits a 450-nm photon, what is its frequency?

62. Vibrations of the hydrogen molecule H_2 can be modeled as a simple harmonic oscillator with the spring constant $k = 1.13 \times 10^3\,\text{N/m}$ and mass $m = 1.67 \times 10^{-27}\,\text{kg}$. (a) What is the vibrational frequency of this molecule? (b) What are the energy and the wavelength of the emitted photon when the molecule makes transition between its third and second excited states?

63. A particle with mass 0.030 kg oscillates back-and-forth on a spring with frequency 4.0 Hz. At the equilibrium position, it has a speed of 0.60 m/s. If the particle is in a state of definite energy, find its energy quantum number.

64. Find the expectation value $\langle x^2 \rangle$ of the square of the position for a quantum harmonic oscillator in the ground state. Note: $\int_{-\infty}^{+\infty} dx x^2 e^{-ax^2} = \sqrt{\pi}(2a^{3/2})^{-1}$.

65. Determine the expectation value of the potential energy for a quantum harmonic oscillator in the ground state. Use this to calculate the expectation value of the kinetic energy.

66. Verify that $\psi_1(x)$ given by **Equation 7.57** is a solution of Schrödinger's equation for the quantum harmonic oscillator.

67. Estimate the ground state energy of the quantum harmonic oscillator by Heisenberg's uncertainty principle. Start by assuming that the product of the uncertainties Δx and Δp is at its minimum. Write Δp in terms of Δx and assume that for the ground state $x \approx \Delta x$ and $p \approx \Delta p$, then write the ground state energy in terms of x. Finally, find the value of x that minimizes the energy and find the minimum of the energy.

68. A mass of 0.250 kg oscillates on a spring with the force constant 110 N/m. Calculate the ground energy level and the separation between the adjacent energy levels. Express the results in joules and in electron-volts. Are quantum effects important?

7.6 The Quantum Tunneling of Particles through Potential Barriers

69. Show that the wave function in (a) **Equation 7.68** satisfies **Equation 7.61**, and (b) **Equation 7.69** satisfies **Equation 7.63**.

70. A 6.0-eV electron impacts on a barrier with height 11.0 eV. Find the probability of the electron to tunnel through the barrier if the barrier width is (a) 0.80 nm and (b) 0.40 nm.

71. A 5.0-eV electron impacts on a barrier of with 0.60 nm. Find the probability of the electron to tunnel through the barrier if the barrier height is (a) 7.0 eV; (b) 9.0 eV; and (c) 13.0 eV.

72. A 12.0-eV electron encounters a barrier of height 15.0 eV. If the probability of the electron tunneling through the barrier is 2.5 %, find its width.

73. A quantum particle with initial kinetic energy 32.0 eV encounters a square barrier with height 41.0 eV and width 0.25 nm. Find probability that the particle tunnels through this barrier if the particle is (a) an electron and, (b) a proton.

74. A simple model of a radioactive nuclear decay assumes that α-particles are trapped inside a well of nuclear potential that walls are the barriers of a finite width 2.0 fm and height 30.0 MeV. Find the tunneling probability across the potential barrier of the wall for α-particles having kinetic energy (a) 29.0 MeV and (b) 20.0 MeV. The mass of the α-particle is $m = 6.64 \times 10^{-27}\,\text{kg}$.

75. A muon, a quantum particle with a mass approximately 200 times that of an electron, is incident on a potential barrier of height 10.0 eV. The kinetic energy of the impacting muon is 5.5 eV and only about 0.10% of the squared amplitude of its incoming wave function filters through the barrier. What is the barrier's width?

76. A grain of sand with mass 1.0 mg and kinetic energy 1.0 J is incident on a potential energy barrier with height 1.000001 J and width 2500 nm. How many grains of sand have to fall on this barrier before, on the average, one passes through?

ADDITIONAL PROBLEMS

77. Show that if the uncertainty in the position of a particle is on the order of its de Broglie's wavelength, then the uncertainty in its momentum is on the order of the value of its momentum.

78. The mass of a ρ-meson is measured to be $770\,\text{MeV}/c^2$ with an uncertainty of $100\,\text{MeV}/c^2$. Estimate the lifetime of this meson.

79. A particle of mass m is confined to a box of width L. If the particle is in the first excited state, what are the probabilities of finding the particle in a region of width 0.020 L around the given point x: (a) $x = 0.25L$; (b) $x = 0.40L$; (c) $x = 0.75L$; and (d) $x = 0.90L$.

80. A particle in a box [0;L] is in the third excited state. What are its most probable positions?

81. A 0.20-kg billiard ball bounces back and forth without losing its energy between the cushions of a 1.5 m long table. (a) If the ball is in its ground state, how many years does it need to get from one cushion to the other? You may compare this time interval to the age of the universe. (b) How much energy is required to make the ball go from its ground state to its first excited state? Compare it with the kinetic energy of the ball moving at 2.0 m/s.

82. Find the expectation value of the position squared when the particle in the box is in its third excited state and the length of the box is L.

83. Consider an infinite square well with wall boundaries $x = 0$ and $x = L$. Show that the function $\psi(x) = A \sin kx$ is the solution to the stationary Schrödinger equation for the particle in a box only if $k = \sqrt{2mE}/\hbar$. Explain why this is an acceptable wave function only if k is an integer multiple of π/L.

84. Consider an infinite square well with wall boundaries $x = 0$ and $x = L$. Explain why the function $\psi(x) = A \cos kx$ is not a solution to the stationary Schrödinger equation for the particle in a box.

85. Atoms in a crystal lattice vibrate in simple harmonic motion. Assuming a lattice atom has a mass of 9.4×10^{-26} kg, what is the force constant of the lattice if a lattice atom makes a transition from the ground state to first excited state when it absorbs a $525\text{-}\mu\text{m}$ photon?

86. A diatomic molecule behaves like a quantum harmonic oscillator with the force constant 12.0 N/m and mass 5.60×10^{-26} kg. (a) What is the wavelength of the emitted photon when the molecule makes the transition from the third excited state to the second excited state? (b) Find the ground state energy of vibrations for this diatomic molecule.

87. An electron with kinetic energy 2.0 MeV encounters a potential energy barrier of height 16.0 MeV and width 2.00 nm. What is the probability that the electron emerges on the other side of the barrier?

88. A beam of mono-energetic protons with energy 2.0 MeV falls on a potential energy barrier of height 20.0 MeV and of width 1.5 fm. What percentage of the beam is transmitted through the barrier?

CHALLENGE PROBLEMS

89. An electron in a long, organic molecule used in a dye laser behaves approximately like a quantum particle in a box with width 4.18 nm. Find the emitted photon when the electron makes a transition from the first excited state to the ground state and from the second excited state to the first excited state.

90. In STM, an elevation of the tip above the surface being scanned can be determined with a great precision, because the tunneling-electron current between surface atoms and the atoms of the tip is extremely sensitive to the variation of the separation gap between them from point to point along the surface. Assuming that the tunneling-electron current is in direct proportion to the tunneling probability and that the tunneling probability is to a good approximation expressed by the exponential function $e^{-2\beta L}$ with $\beta = 10.0/\text{nm}$, determine the ratio of the tunneling current when the tip is 0.500 nm above the surface to the current when the tip is 0.515 nm above the surface.

91. If STM is to detect surface features with local heights of about 0.00200 nm, what percent change in tunneling-electron current must the STM electronics be able to detect? Assume that the tunneling-electron current has characteristics given in the preceding problem.

92. Use Heisenberg's uncertainty principle to estimate the ground state energy of a particle oscillating on an spring with angular frequency, $\omega = \sqrt{k/m}$, where k is the spring constant and m is the mass.

93. Suppose an infinite square well extends from $-L/2$ to $+L/2$. Solve the time-independent Schrödinger's equation to find the allowed energies and stationary states of a particle with mass m that is confined to this well. Then show that these solutions can be obtained by making the coordinate transformation $x' = x - L/2$ for the solutions obtained for the well extending between 0 and L.

94. A particle of mass m confined to a box of width L is in its first excited state $\psi_2(x)$. (a) Find its average position (which is the expectation value of the position). (b) Where is the particle most likely to be found?

8 | ATOMIC STRUCTURE

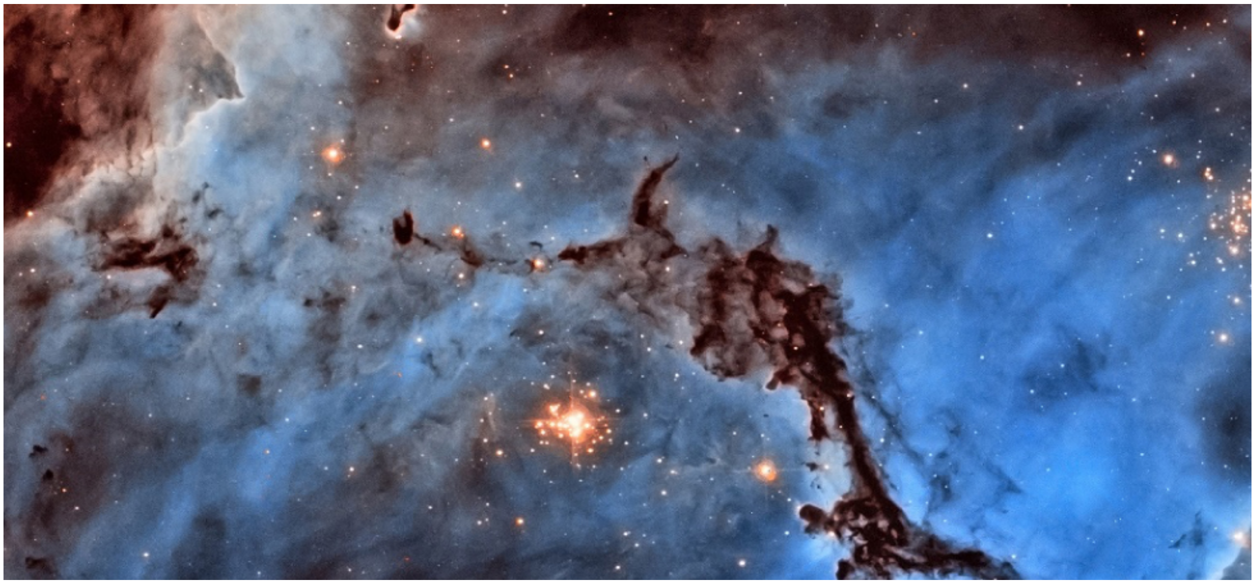


Figure 8.1 NGC1763 is an emission nebula in the Large Magellanic Cloud, which is a satellite galaxy to our Milky Way Galaxy. The colors we see can be explained by applying the ideas of quantum mechanics to atomic structure. (credit: NASA, ESA, and Josh Lake)

Chapter Outline

Introduction

In this chapter, we use quantum mechanics to study the structure and properties of atoms. This study introduces ideas and concepts that are necessary to understand more complex systems, such as molecules, crystals, and metals. As we deepen our understanding of atoms, we build on things we already know, such as Rutherford's nuclear model of the atom, Bohr's model of the hydrogen atom, and de Broglie's wave hypothesis.

Figure 8.1 is NGC1763, an emission nebula in the small galaxy known as the Large Magellanic Cloud, which is a satellite of the Milky Way Galaxy. Ultraviolet light from hot stars ionizes the hydrogen atoms in the nebula. As protons and electrons recombine, radiation of different frequencies is emitted. The details of this process can be correctly predicted by quantum mechanics and are examined in this chapter.

8.1 | The Hydrogen Atom

Learning Objectives

By the end of this section, you will be able to:

- Describe the hydrogen atom in terms of wave function, probability density, total energy, and orbital angular momentum
- Identify the physical significance of each of the quantum numbers (n, l, m) of the hydrogen atom
- Distinguish between the Bohr and Schrödinger models of the atom
- Use quantum numbers to calculate important information about the hydrogen atom

The hydrogen atom is the simplest atom in nature and, therefore, a good starting point to study atoms and atomic structure. The hydrogen atom consists of a single negatively charged electron that moves about a positively charged proton (**Figure 8.2**). In Bohr's model, the electron is pulled around the proton in a perfectly circular orbit by an attractive Coulomb force. The proton is approximately 1800 times more massive than the electron, so the proton moves very little in response to the force on the proton by the electron. (This is analogous to the Earth-Sun system, where the Sun moves very little in response to the force exerted on it by Earth.) An explanation of this effect using Newton's laws is given in **Photons and Matter Waves**.

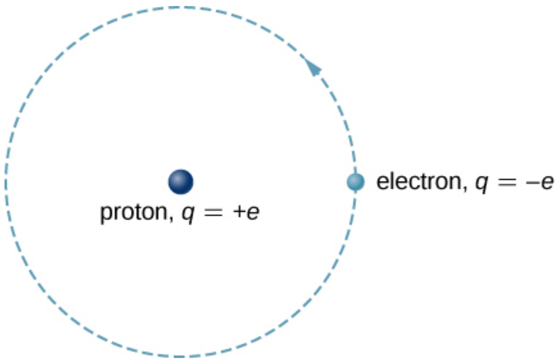


Figure 8.2 A representation of the Bohr model of the hydrogen atom.

With the assumption of a fixed proton, we focus on the motion of the electron.

In the electric field of the proton, the potential energy of the electron is

$$U(r) = -k\frac{e^2}{r}, \tag{8.1}$$

where $k = 1/4\pi\varepsilon_0$ and r is the distance between the electron and the proton. As we saw earlier, the force on an object is equal to the negative of the gradient (or slope) of the potential energy function. For the special case of a hydrogen atom, the force between the electron and proton is an attractive Coulomb force.

Notice that the potential energy function $U(r)$ does not vary in time. As a result, Schrödinger's equation of the hydrogen atom reduces to two simpler equations: one that depends only on space (x, y, z) and another that depends only on time (t). (The separation of a wave function into space- and time-dependent parts for time-independent potential energy functions is discussed in **Quantum Mechanics**.) We are most interested in the space-dependent equation:

$$\frac{-\hbar^2}{2m_e}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) - k\frac{e^2}{r}\psi = E\psi, \tag{8.2}$$

where $\psi = \psi(x,\ y,\ z)$ is the three-dimensional wave function of the electron, m_e is the mass of the electron, and E is the total energy of the electron. Recall that the total wave function $\Psi(x,\ y,\ z,\ t)$, is the product of the space-dependent wave function $\psi = \psi(x,\ y,\ z)$ and the time-dependent wave function $\varphi = \varphi(t)$.

In addition to being time-independent, $U(r)$ is also spherically symmetrical. This suggests that we may solve Schrödinger's equation more easily if we express it in terms of the spherical coordinates (r, θ, ϕ) instead of rectangular coordinates $(x,\ y,\ z)$. A spherical coordinate system is shown in **Figure 8.3**. In spherical coordinates, the variable r is the radial coordinate, θ is the polar angle (relative to the vertical z-axis), and ϕ is the azimuthal angle (relative to the x-axis). The relationship between spherical and rectangular coordinates is $x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta.$

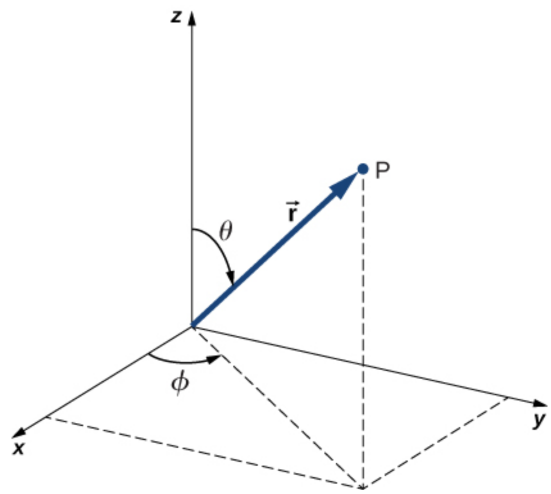


Figure 8.3 The relationship between the spherical and rectangular coordinate systems.

The factor $r\sin\theta$ is the magnitude of a vector formed by the projection of the polar vector onto the xy-plane. Also, the coordinates of x and y are obtained by projecting this vector onto the x- and y-axes, respectively. The inverse transformation gives

$$r = \sqrt{x^2 + y^2 + z^2},\quad \theta = \cos^{-1}\left(\frac{z}{r}\right),\quad \phi = \cos^{-1}\left(\frac{x}{\sqrt{x^2 + y^2}}\right).$$

Schrödinger's wave equation for the hydrogen atom in spherical coordinates is discussed in more advanced courses in modern physics, so we do not consider it in detail here. However, due to the spherical symmetry of $U(r)$, this equation reduces to three simpler equations: one for each of the three coordinates $(r,\ \theta,\ \text{and}\ \phi)$. Solutions to the time-independent wave function are written as a product of three functions:

$$\psi(r,\ \theta,\ \phi) = R(r)\Theta(\theta)\Phi(\phi),$$

where R is the radial function dependent on the radial coordinate r only; Θ is the polar function dependent on the polar coordinate θ only; and Φ is the phi function of ϕ only. Valid solutions to Schrödinger's equation $\psi(r,\ \theta,\ \phi)$ are labeled by the quantum numbers n, l, and m.

n : principal quantum number
l : angular momentum quantum number
m : angular momentum projection quantum number

(The reasons for these names will be explained in the next section.) The radial function R depends only on n and l; the polar function Θ depends only on l and m; and the phi function Φ depends only on m. The dependence of each function on quantum numbers is indicated with subscripts:

$$\psi_{nlm}(r,\ \theta,\ \phi) = R_{nl}(r)\Theta_{lm}(\theta)\Phi_m(\phi).$$

Not all sets of quantum numbers (*n*, *l*, *m*) are possible. For example, the orbital angular quantum number *l* can never be greater or equal to the principal quantum number $n(l < n)$. Specifically, we have

$$\begin{aligned} n &= 1, 2, 3, \ldots \\ l &= 0, 1, 2, \ldots, (n-1) \\ m &= -l, (-l+1), \ldots, 0, \ldots, (+l-1), +l \end{aligned}$$

Notice that for the ground state, $n = 1$, $l = 0$, and $m = 0$. In other words, there is only one quantum state with the wave function for $n = 1$, and it is ψ_{100}. However, for $n = 2$, we have

$$\begin{aligned} l = 0, &\quad m = 0 \\ l = 1, &\quad m = -1, 0, 1. \end{aligned}$$

Therefore, the allowed states for the $n = 2$ state are ψ_{200}, ψ_{21-1}, ψ_{210}, and ψ_{211}. Example wave functions for the hydrogen atom are given in **Table 8.1**. Note that some of these expressions contain the letter *i*, which represents $\sqrt{-1}$. When probabilities are calculated, these complex numbers do not appear in the final answer.

$n = 1, l = 0, m_l = 0$	$\psi_{100} = \frac{1}{\sqrt{\pi}} \frac{1}{a_0{}^{3/2}} e^{-r/a_0}$
$n = 2, l = 0, m_l = 0$	$\psi_{200} = \frac{1}{4\sqrt{2\pi}} \frac{1}{a_0{}^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$
$n = 2, l = 1, m_l = -1$	$\psi_{21-1} = \frac{1}{8\sqrt{\pi}} \frac{1}{a_0{}^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \sin\theta e^{-i\phi}$
$n = 2, l = 1, m_l = 0$	$\psi_{210} = \frac{1}{4\sqrt{2\pi}} \frac{1}{a_0{}^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \cos\theta$
$n = 2, l = 1, m_l = 1$	$\psi_{211} = \frac{1}{8\sqrt{\pi}} \frac{1}{a_0{}^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \sin\theta e^{i\phi}$

Table 8.1 Wave Functions of the Hydrogen Atom

Physical Significance of the Quantum Numbers

Each of the three quantum numbers of the hydrogen atom (*n*, *l*, *m*) is associated with a different physical quantity. The **principal quantum number** *n* is associated with the total energy of the electron, E_n. According to Schrödinger's equation:

$$E_n = -\left(\frac{m_e k^2 e^4}{2\hbar^2}\right)\left(\frac{1}{n^2}\right) = -E_0\left(\frac{1}{n^2}\right), \tag{8.3}$$

where $E_0 = -13.6\,\text{eV}$. Notice that this expression is identical to that of Bohr's model. As in the Bohr model, the electron in a particular state of energy does not radiate.

Example 8.1

How Many Possible States?

For the hydrogen atom, how many possible quantum states correspond to the principal number $n = 3$? What are the energies of these states?

Strategy

For a hydrogen atom of a given energy, the number of allowed states depends on its orbital angular momentum. We can count these states for each value of the principal quantum number, $n = 1, 2, 3$. However, the total energy depends on the principal quantum number only, which means that we can use **Equation 8.3** and the number of states counted.

Solution

If $n = 3$, the allowed values of l are 0, 1, and 2. If $l = 0$, $m = 0$ (1 state). If $l = 1$, $m = -1, 0, +1$ (3 states); and if $l = 2$, $m = -2, -1, 0, +1, +2$ (5 states). In total, there are $1 + 3 + 5 = 9$ allowed states. Because the total energy depends only on the principal quantum number, $n = 3$, the energy of each of these states is

$$E_{n3} = -E_0\left(\frac{1}{n^2}\right) = \frac{-13.6\text{ eV}}{9} = -1.51\text{ eV}.$$

Significance

An electron in a hydrogen atom can occupy many different angular momentum states with the very same energy. As the orbital angular momentum increases, the number of the allowed states with the same energy increases.

The **angular momentum orbital quantum number** *l* is associated with the orbital angular momentum of the electron in a hydrogen atom. Quantum theory tells us that when the hydrogen atom is in the state ψ_{nlm}, the magnitude of its orbital angular momentum is

$$L = \sqrt{l(l+1)}\hbar, \tag{8.4}$$

where

$$l = 0, 1, 2, \ldots, (n-1).$$

This result is slightly different from that found with Bohr's theory, which quantizes angular momentum according to the rule $L = n$, where $n = 1, 2, 3, \ldots$.

Quantum states with different values of orbital angular momentum are distinguished using spectroscopic notation (**Table 8.2**). The designations *s*, *p*, *d*, and *f* result from early historical attempts to classify atomic spectral lines. (The letters stand for sharp, principal, diffuse, and fundamental, respectively.) After *f*, the letters continue alphabetically.

The ground state of hydrogen is designated as the 1*s* state, where "1" indicates the energy level $(n = 1)$ and "*s*" indicates the orbital angular momentum state ($l = 0$). When $n = 2$, l can be either 0 or 1. The $n = 2$, $l = 0$ state is designated "2*s*." The $n = 2$, $l = 1$ state is designated "2*p*." When $n = 3$, l can be 0, 1, or 2, and the states are 3*s*, 3*p*, and 3*d*, respectively. Notation for other quantum states is given in **Table 8.3**.

The **angular momentum projection quantum number** *m* is associated with the azimuthal angle ϕ (see **Figure 8.3**) and is related to the *z*-component of orbital angular momentum of an electron in a hydrogen atom. This component is given by

$$L_z = m\hbar, \tag{8.5}$$

where

$$m = -l, -l+1, \ldots, 0, \ldots, +l-1, l.$$

The *z*-component of angular momentum is related to the magnitude of angular momentum by

$$L_z = L\cos\theta, \tag{8.6}$$

where θ is the angle between the angular momentum vector and the *z*-axis. Note that the direction of the *z*-axis is determined by experiment—that is, along any direction, the experimenter decides to measure the angular momentum. For example, the *z*-direction might correspond to the direction of an external magnetic field. The relationship between L_z and L is given in **Figure 8.4**.

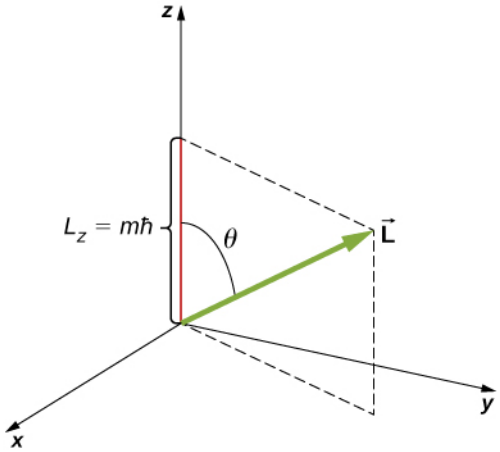


Figure 8.4 The *z*-component of angular momentum is quantized with its own quantum number *m*.

Orbital Quantum Number *l*	Angular Momentum	State	Spectroscopic Name
0	0	*s*	Sharp
1	$\sqrt{2}h$	*p*	Principal
2	$\sqrt{6}h$	*d*	Diffuse
3	$\sqrt{12}h$	*f*	Fundamental
4	$\sqrt{20}h$	*g*	
5	$\sqrt{30}h$	*h*	

Table 8.2 Spectroscopic Notation and Orbital Angular Momentum

	$l = 0$	$l = 1$	$l = 2$	$l = 3$	$l = 4$	$l = 5$
$n = 1$	1*s*					
$n = 2$	2*s*	2*p*				
$n = 3$	3*s*	3*p*	3*d*			
$n = 4$	4*s*	4*p*	4*d*	4*f*		
$n = 5$	5*s*	5*p*	5*d*	5*f*	5*g*	
$n = 6$	6*s*	6*p*	6*d*	6*f*	6*g*	6*h*

Table 8.3 Spectroscopic Description of Quantum States

The quantization of L_z is equivalent to the quantization of θ. Substituting $\sqrt{l(l+1)}\hbar$ for L and m for L_z into this equation, we find

$$m\hbar = \sqrt{l(l+1)}\hbar \cos\theta. \quad \textbf{(8.7)}$$

Thus, the angle θ is quantized with the particular values

$$\theta = \cos^{-1}\left(\frac{m}{\sqrt{l(l+1)}}\right). \quad \textbf{(8.8)}$$

Notice that both the polar angle (θ) and the projection of the angular momentum vector onto an arbitrary *z*-axis (L_z) are quantized.

The quantization of the polar angle for the $l = 3$ state is shown in **Figure 8.5**. The orbital angular momentum vector lies somewhere on the surface of a cone with an opening angle θ relative to the *z*-axis (unless $m = 0,$ in which case $\theta = 90°$ and the vector points are perpendicular to the *z*-axis).

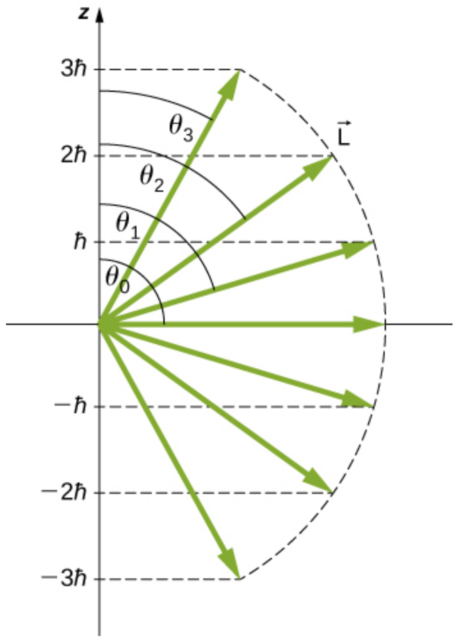


Figure 8.5 The quantization of orbital angular momentum. Each vector lies on the surface of a cone with axis along the *z*-axis.

A detailed study of angular momentum reveals that we cannot know all three components simultaneously. In the previous section, the *z*-component of orbital angular momentum has definite values that depend on the quantum number *m*. This implies that we cannot know both *x*- and *y*-components of angular momentum, L_x and L_y, with certainty. As a result, the precise direction of the orbital angular momentum vector is unknown.

Example 8.2

What Are the Allowed Directions?

Calculate the angles that the angular momentum vector $\vec{\mathbf{L}}$ can make with the *z*-axis for $l = 1$, as shown in **Figure 8.6**.

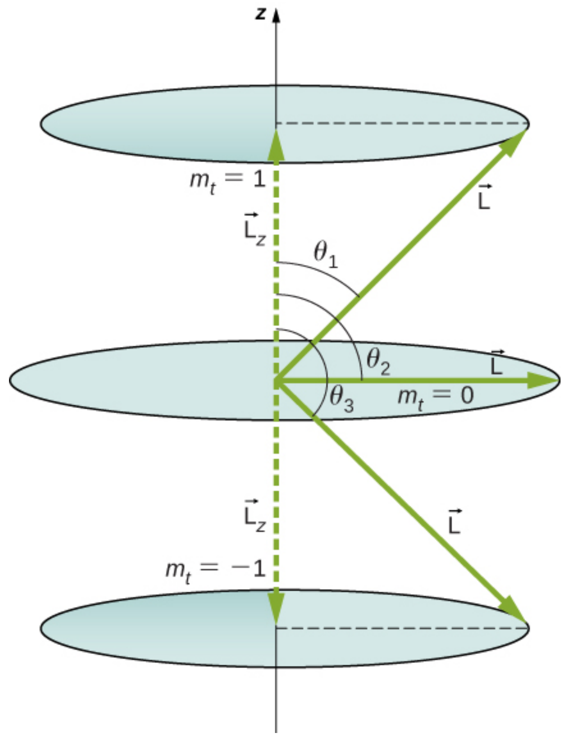


Figure 8.6 The component of a given angular momentum along the *z*-axis (defined by the direction of a magnetic field) can have only certain values. These are shown here for $l = 1$, for which $m = -1,\ 0,\ \text{and} + 1.$ The direction of $\vec{\mathbf{L}}$ is quantized in the sense that it can have only certain angles relative to the *z*-axis.

Strategy

The vectors $\vec{\mathbf{L}}$ and $\vec{\mathbf{L}}_z$ (in the *z*-direction) form a right triangle, where $\vec{\mathbf{L}}$ is the hypotenuse and $\vec{\mathbf{L}}_z$ is the adjacent side. The ratio of L_z to $|\vec{\mathbf{L}}|$ is the cosine of the angle of interest. The magnitudes $L = \left|\vec{\mathbf{L}}\right|$ and L_z are given by

$$L = \sqrt{l(l+1)}\hbar \text{ and } L_z = m\hbar.$$

Solution

We are given $l = 1$, so *ml* can be $+1,\ 0,\ \text{or} \ -1.$ Thus, *L* has the value given by

$$L = \sqrt{l(l+1)}\hbar = \sqrt{2}\hbar.$$

The quantity L_z can have three values, given by $L_z = m_l \hbar$.

$$L_z = m_l \hbar = \begin{cases} \hbar, & m_l = +1 \\ 0, & m_l = 0 \\ -\hbar, & m_l = -1 \end{cases}$$

As you can see in **Figure 8.6**, $\cos\theta = L_z/L,$ so for $m = +1$, we have

$$\cos\theta_1 = \frac{L_Z}{L} = \frac{\hbar}{\sqrt{2}\hbar} = \frac{1}{\sqrt{2}} = 0.707.$$

Thus,

$$\theta_1 = \cos^{-1} 0.707 = 45.0°.$$

Similarly, for $m = 0$, we find $\cos\theta_2 = 0$; this gives

$$\theta_2 = \cos^{-1} 0 = 90.0°.$$

Then for $m_l = -1$:

$$\cos\theta_3 = \frac{L_Z}{L} = \frac{-\hbar}{\sqrt{2}\hbar} = -\frac{1}{\sqrt{2}} = -0.707,$$

so that

$$\theta_3 = \cos^{-1}(-0.707) = 135.0°.$$

Significance

The angles are consistent with the figure. Only the angle relative to the *z*-axis is quantized. *L* can point in any direction as long as it makes the proper angle with the *z*-axis. Thus, the angular momentum vectors lie on cones, as illustrated. To see how the correspondence principle holds here, consider that the smallest angle (θ_1 in the example) is for the maximum value of m_l, namely $m_l = l$. For that smallest angle,

$$\cos\theta = \frac{L_z}{L} = \frac{l}{\sqrt{l(l+1)}},$$

which approaches 1 as *l* becomes very large. If $\cos\theta = 1$, then $\theta = 0°$. Furthermore, for large *l*, there are many values of m_l, so that all angles become possible as *l* gets very large.

8.1 Check Your Understanding Can the magnitude of L_z ever be equal to *L*?

Using the Wave Function to Make Predictions

As we saw earlier, we can use quantum mechanics to make predictions about physical events by the use of probability statements. It is therefore proper to state, "An electron is located within this volume with this probability at this time," but not, "An electron is located at the position (*x*, *y*, *z*) at this time." To determine the probability of finding an electron in a hydrogen atom in a particular region of space, it is necessary to integrate the probability density $|\psi_{nlm}|^2$ over that region:

$$\text{Probability} = \int_{\text{volume}} |\psi_{nlm}|^2 dV, \tag{8.9}$$

where *dV* is an infinitesimal volume element. If this integral is computed for all space, the result is 1, because the probability of the particle to be located *somewhere* is 100% (the normalization condition). In a more advanced course on modern physics, you will find that $|\psi_{nlm}|^2 = \psi^*_{nlm}\psi_{nlm}$, where ψ^*_{nlm} is the complex conjugate. This eliminates the occurrences of $i = \sqrt{-1}$ in the above calculation.

Consider an electron in a state of zero angular momentum ($l = 0$). In this case, the electron's wave function depends only on the radial coordinate *r*. (Refer to the states ψ_{100} and ψ_{200} in **Table 8.1**.) The infinitesimal volume element corresponds to a spherical shell of radius *r* and infinitesimal thickness *dr*, written as

$$dV = 4\pi r^2 dr. \tag{8.10}$$

The probability of finding the electron in the region *r* to $r + dr$ ("at approximately *r*") is

$$P(r)dr = |\psi_{n00}|^2 4\pi r^2 dr. \tag{8.11}$$

Here $P(r)$ is called the **radial probability density function** (a probability per unit length). For an electron in the ground state of hydrogen, the probability of finding an electron in the region r to $r + dr$ is

$$|\psi_{n00}|^2 4\pi r^2 dr = (4/a_0^3)r^2 \exp(-2r/a_0)dr, \tag{8.12}$$

where $a_0 = 0.5$ angstroms. The radial probability density function $P(r)$ is plotted in **Figure 8.7**. The area under the curve between any two radial positions, say r_1 and r_2, gives the probability of finding the electron in that radial range. To find the most probable radial position, we set the first derivative of this function to zero ($dP/dr = 0$) and solve for r. The most probable radial position is not equal to the average or expectation value of the radial position because $|\psi_{n00}|^2$ is not symmetrical about its peak value.

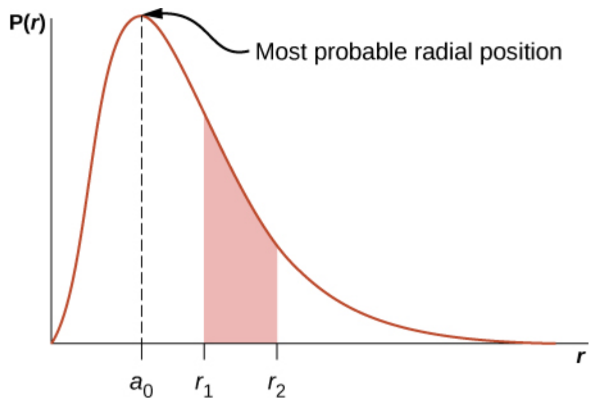


Figure 8.7 The radial probability density function for the ground state of hydrogen.

If the electron has orbital angular momentum ($l \neq 0$), then the wave functions representing the electron depend on the angles θ and ϕ; that is, $\psi_{nlm} = \psi_{nlm}(r, \theta, \phi)$. Atomic orbitals for three states with $n = 2$ and $l = 1$ are shown in **Figure 8.8**. An **atomic orbital** is a region in space that encloses a certain percentage (usually 90%) of the electron probability. (Sometimes atomic orbitals are referred to as "clouds" of probability.) Notice that these distributions are pronounced in certain directions. This directionality is important to chemists when they analyze how atoms are bound together to form molecules.

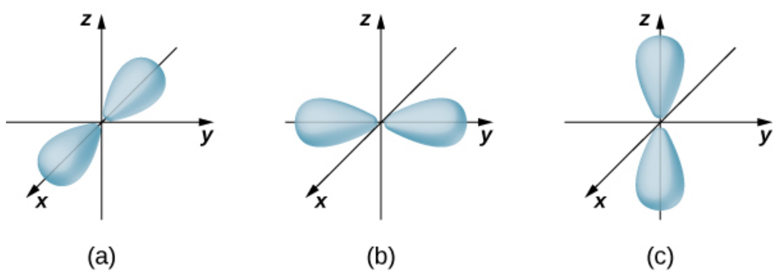


Figure 8.8 The probability density distributions for three states with $n = 2$ and $l = 1$. The distributions are directed along the (a) x-axis, (b) y-axis, and (c) z-axis.

A slightly different representation of the wave function is given in **Figure 8.9**. In this case, light and dark regions indicate locations of relatively high and low probability, respectively. In contrast to the Bohr model of the hydrogen atom, the electron does not move around the proton nucleus in a well-defined path. Indeed, the uncertainty principle makes it impossible to know how the electron gets from one place to another.

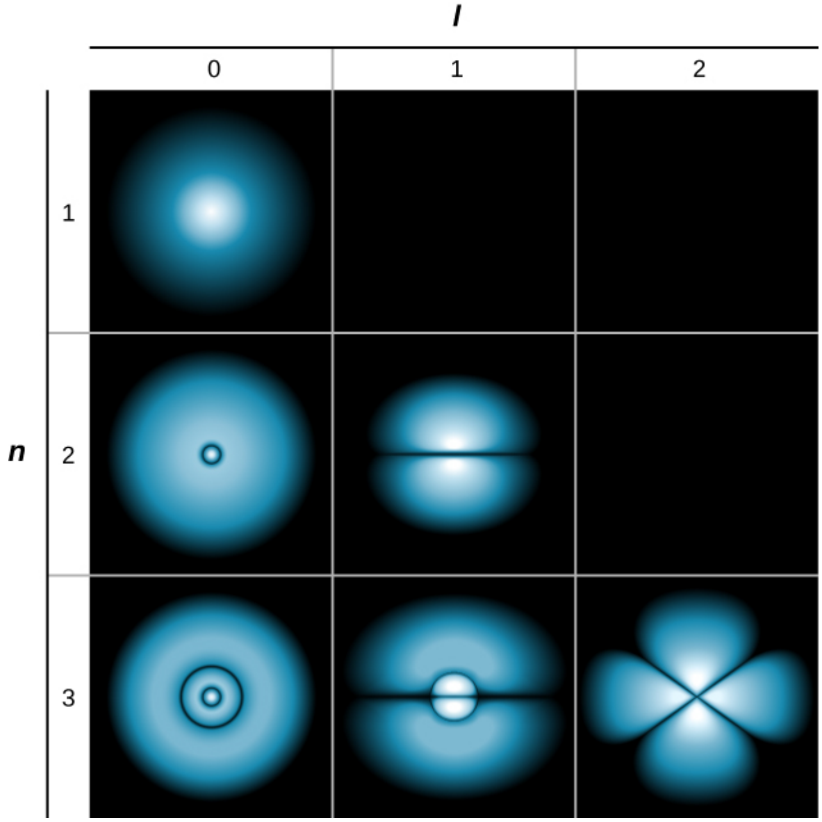


Figure 8.9 Probability clouds for the electron in the ground state and several excited states of hydrogen. The probability of finding the electron is indicated by the shade of color; the lighter the coloring, the greater the chance of finding the electron.

8.2 | Orbital Magnetic Dipole Moment of the Electron

Learning Objectives

By the end of this section, you will be able to:

- Explain why the hydrogen atom has magnetic properties
- Explain why the energy levels of a hydrogen atom associated with orbital angular momentum are split by an external magnetic field
- Use quantum numbers to calculate the magnitude and direction of the orbital magnetic dipole moment of a hydrogen atom

In Bohr's model of the hydrogen atom, the electron moves in a circular orbit around the proton. The electron passes by a particular point on the loop in a certain time, so we can calculate a current $I = Q/t$. An electron that orbits a proton in a hydrogen atom is therefore analogous to current flowing through a circular wire (**Figure 8.10**). In the study of magnetism, we saw that a current-carrying wire produces magnetic fields. It is therefore reasonable to conclude that the hydrogen atom produces a magnetic field and interacts with other magnetic fields.

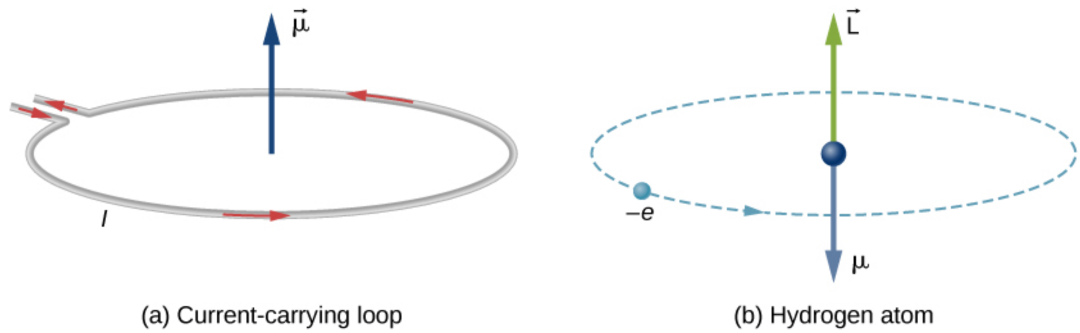


Figure 8.10 (a) Current flowing through a circular wire is analogous to (b) an electron that orbits a proton in a hydrogen atom.

The **orbital magnetic dipole moment** is a measure of the strength of the magnetic field produced by the orbital angular momentum of an electron. From **m58743 (http://cnx.org/content/m58743/latest/#fs-id1171360245659)** , the magnitude of the orbital magnetic dipole moment for a current loop is

$$\mu = IA, \tag{8.13}$$

where I is the current and A is the area of the loop. (For brevity, we refer to this as the magnetic moment.) The current I associated with an electron in orbit about a proton in a hydrogen atom is

$$I = \frac{e}{T}, \tag{8.14}$$

where e is the magnitude of the electron charge and T is its orbital period. If we assume that the electron travels in a perfectly circular orbit, the orbital period is

$$T = \frac{2\pi r}{v}, \tag{8.15}$$

where r is the radius of the orbit and v is the speed of the electron in its orbit. Given that the area of a circle is πr^2 , the absolute magnetic moment is

$$\mu = IA = \frac{e}{\left(\frac{2\pi r}{v}\right)}\pi r^2 = \frac{evr}{2}. \tag{8.16}$$

It is helpful to express the magnetic momentum μ in terms of the orbital angular momentum ($\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$). Because the electron orbits in a circle, the position vector $\vec{\mathbf{r}}$ and the momentum vector $\vec{\mathbf{p}}$ form a right angle. Thus, the magnitude of the orbital angular momentum is

$$L = \left|\vec{\mathbf{L}}\right| = \left|\vec{\mathbf{r}} \times \vec{\mathbf{p}}\right| = rp\sin\theta = rp = rmv. \tag{8.17}$$

Combining these two equations, we have

$$\mu = \left(\frac{e}{2m_e}\right)L. \tag{8.18}$$

In full vector form, this expression is written as

$$\vec{\boldsymbol{\mu}} = -\left(\frac{e}{2m_e}\right)\vec{\boldsymbol{L}}. \tag{8.19}$$

The negative sign appears because the electron has a negative charge. Notice that the direction of the magnetic moment of the electron is antiparallel to the orbital angular momentum, as shown in **Figure 8.10**(b). In the Bohr model of the atom, the relationship between $\vec{\mu}$ and $\vec{\mathbf{L}}$ in **Equation 8.19** is independent of the radius of the orbit.

The magnetic moment μ can also be expressed in terms of the orbital angular quantum number l. Combining **Equation 8.18** and **Equation 8.15**, the magnitude of the magnetic moment is

$$\mu = \left(\frac{e}{2m_e}\right)L = \left(\frac{e}{2m_e}\right)\sqrt{l(l+1)}\hbar = \mu_B\sqrt{l(l+1)}. \tag{8.20}$$

The z-component of the magnetic moment is

$$\mu_z = -\left(\frac{e}{2m_e}\right)L_z = -\left(\frac{e}{2m_e}\right)m\hbar = -\mu_B m. \tag{8.21}$$

The quantity μ_B is a fundamental unit of magnetism called the **Bohr magneton**, which has the value 9.3×10^{-24} joule/tesla (J/T) or 5.8×10^{-5} eV/T. Quantization of the magnetic moment is the result of quantization of the orbital angular momentum.

As we will see in the next section, the total magnetic dipole moment of the hydrogen atom is due to both the orbital motion of the electron and its intrinsic spin. For now, we ignore the effect of electron spin.

Example 8.3

Orbital Magnetic Dipole Moment

What is the magnitude of the orbital dipole magnetic moment μ of an electron in the hydrogen atom in the (a) *s* state, (b) *p* state, and (c) *d* state? (Assume that the spin of the electron is zero.)

Strategy

The magnetic momentum of the electron is related to its orbital angular momentum L. For the hydrogen atom, this quantity is related to the orbital angular quantum number l. The states are given in spectroscopic notation, which relates a letter (*s*, *p*, *d*, etc.) to a quantum number.

Solution

The magnitude of the magnetic moment is given in **Equation 8.20**:

$$\mu = \left(\frac{e}{2m_e}\right)L = \left(\frac{e}{2m_e}\right)\sqrt{l(l+1)}\hbar = \mu_B\sqrt{l(l+1)}.$$

a. For the *s* state, $l = 0$ so we have $\mu = 0$ and $\mu_z = 0$.

b. For the *p* state, $l = 1$ and we have

$$\mu = \mu_B\sqrt{1(1+1)} = \sqrt{2}\mu_B$$
$$\mu_z = -\mu_B m, \text{ where } m = (-1, 0, 1), \text{ so}$$
$$\mu_z = \mu_B, 0, -\mu_B.$$

c. For the *d* state, $l = 2$ and we obtain

$$\mu = \mu_B\sqrt{2(2+1)} = \sqrt{6}\mu_B$$
$$\mu_z = -\mu_B m, \text{ where } m = (-2, -1, 0, 1, 2), \text{ so}$$
$$\mu_z = 2\mu_B, \mu_B, 0, -\mu_B, -2\mu_B.$$

Significance

In the *s* state, there is no orbital angular momentum and therefore no magnetic moment. This does not mean that the electron is at rest, just that the overall motion of the electron does not produce a magnetic field. In the *p* state, the electron has a magnetic moment with three possible values for the z-component of this magnetic moment; this means that magnetic moment can point in three different polar directions—each antiparallel to the orbital angular momentum vector. In the *d* state, the electron has a magnetic moment with five possible values for the z-component of this magnetic moment. In this case, the magnetic moment can point in five different polar directions.

A hydrogen atom has a magnetic field, so we expect the hydrogen atom to interact with an external magnetic field—such as the push and pull between two bar magnets. From **m58743 (http://cnx.org/content/m58743/latest/#fs-id1171360288680)** , we know that when a current loop interacts with an external magnetic field $\vec{\mathbf{B}}$, it experiences a torque given by

$$\vec{\tau} = I\left(\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right) = \vec{\mu} \times \vec{\mathbf{B}}, \tag{8.22}$$

where I is the current, $\vec{\mathbf{A}}$ is the area of the loop, $\vec{\mu}$ is the magnetic moment, and $\vec{\mathbf{B}}$ is the external magnetic field. This torque acts to rotate the magnetic moment vector of the hydrogen atom to align with the external magnetic field. Because mechanical work is done by the external magnetic field on the hydrogen atom, we can talk about energy transformations in the atom. The potential energy of the hydrogen atom associated with this magnetic interaction is given by **Equation 8.23**:

$$U = -\vec{\mu} \cdot \vec{B}. \tag{8.23}$$

If the magnetic moment is antiparallel to the external magnetic field, the potential energy is large, but if the magnetic moment is parallel to the field, the potential energy is small. Work done on the hydrogen atom to rotate the atom's magnetic moment vector in the direction of the external magnetic field is therefore associated with a drop in potential energy. The energy of the system is conserved, however, because a drop in potential energy produces radiation (the emission of a photon). These energy transitions are quantized because the magnetic moment can point in only certain directions.

If the external magnetic field points in the positive *z*-direction, the potential energy associated with the orbital magnetic dipole moment is

$$U(\theta) = -\mu B \cos\theta = -\mu_z B = -(-\mu_B m)B = m\mu_B B, \tag{8.24}$$

where μ_B is the Bohr magneton and m is the angular momentum projection quantum number (or **magnetic orbital quantum number**), which has the values

$$m = -l, -l+1, \ldots, 0, \ldots, l-1, l. \tag{8.25}$$

For example, in the $l = 1$ electron state, the total energy of the electron is split into three distinct energy levels corresponding to $U = -\mu_B B, 0, \mu_B B$.

The splitting of energy levels by an external magnetic field is called the **Zeeman effect**. Ignoring the effects of electron spin, transitions from the $l = 1$ state to a common lower energy state produce three closely spaced spectral lines (**Figure 8.11**, left column). Likewise, transitions from the $l = 2$ state produce five closely spaced spectral lines (right column). The separation of these lines is proportional to the strength of the external magnetic field. This effect has many applications. For example, the splitting of lines in the hydrogen spectrum of the Sun is used to determine the strength of the Sun's magnetic field. Many such magnetic field measurements can be used to make a map of the magnetic activity at the Sun's surface called a **magnetogram** (**Figure 8.12**).

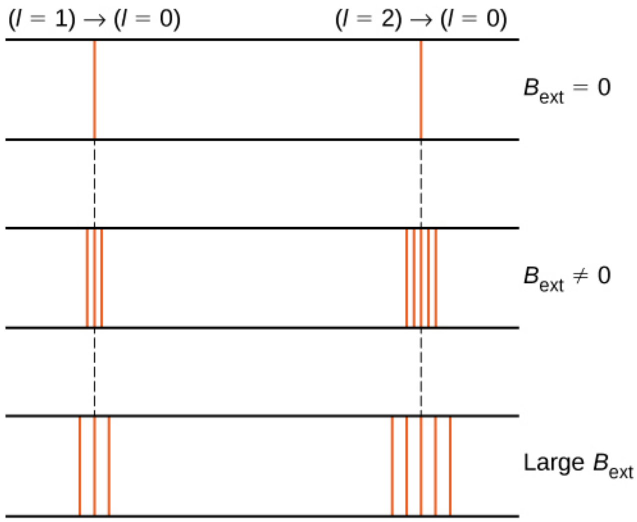


Figure 8.11 The Zeeman effect refers to the splitting of spectral lines by an external magnetic field. In the left column, the energy splitting occurs due to transitions from the state $(n = 2,\ l = 1)$ to a lower energy state; and in the right column, energy splitting occurs due to transitions from the state $(n = 2, l = 2)$ to a lower-energy state. The separation of these lines is proportional to the strength of the external magnetic field.

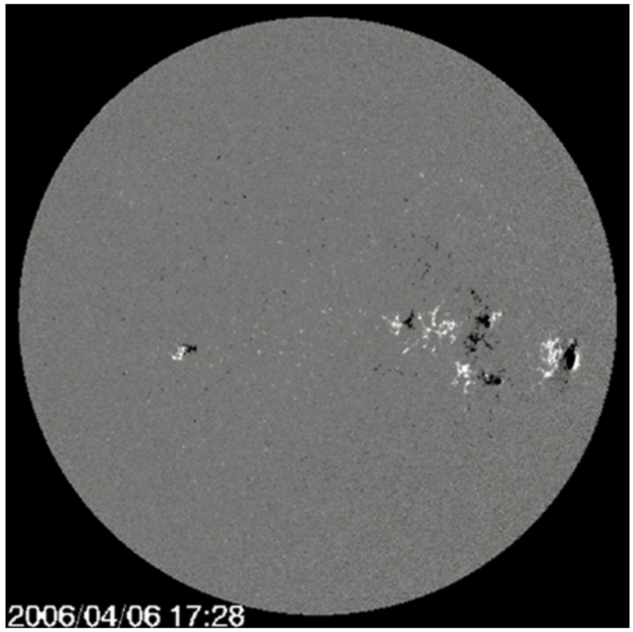


Figure 8.12 A magnetogram of the Sun. The bright and dark spots show significant magnetic activity at the surface of the Sun.

8.3 | Electron Spin

Learning Objectives

By the end of this section, you will be able to:

- Express the state of an electron in a hydrogen atom in terms of five quantum numbers
- Use quantum numbers to calculate the magnitude and direction of the spin and magnetic moment of an electron
- Explain the fine and hyperfine structure of the hydrogen spectrum in terms of magnetic interactions inside the hydrogen atom

In this section, we consider the effects of electron spin. Spin introduces two additional quantum numbers to our model of the hydrogen atom. Both were discovered by looking at the fine structure of atomic spectra. Spin is a fundamental characteristic of all particles, not just electrons, and is analogous to the intrinsic spin of extended bodies about their own axes, such as the daily rotation of Earth.

Spin is quantized in the same manner as orbital angular momentum. It has been found that the magnitude of the intrinsic spin angular momentum S of an electron is given by

$$S = \sqrt{s(s+1)}\hbar, \tag{8.26}$$

where s is defined to be the **spin quantum number**. This is similar to the quantization of L given in **Equation 8.4**, except that the only value allowed for s for an electron is $s = 1/2$. The electron is said to be a "spin-half particle." The **spin projection quantum number** m_s is associated with the z-components of spin, expressed by

$$S_z = m_s \hbar. \tag{8.27}$$

In general, the allowed quantum numbers are

$$m_s = -s, -s+1, \ldots, 0, \ldots, +s-1, s. \tag{8.28}$$

For the special case of an electron ($s = 1/2$),

$$m_s = -\frac{1}{2}, \frac{1}{2}. \tag{8.29}$$

Directions of intrinsic spin are quantized, just as they were for orbital angular momentum. The $m_s = -1/2$ state is called the "spin-down" state and has a z-component of spin, $s_z = -1/2$; the $m_s = +1/2$ state is called the "spin-up" state and has a z-component of spin, $s_z = +1/2$. These states are shown in **Figure 8.13**.

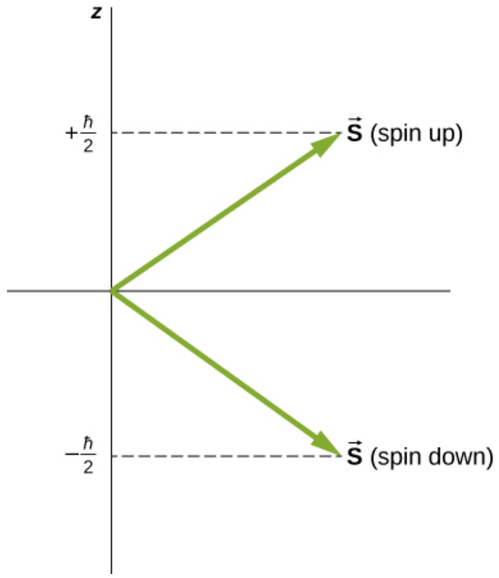


Figure 8.13 The two possible states of electron spin.

The intrinsic magnetic dipole moment of an electron μ_e can also be expressed in terms of the spin quantum number. In analogy to the orbital angular momentum, the magnitude of the electron magnetic moment is

$$\mu_s = \left(\frac{e}{2m_e}\right)S. \tag{8.30}$$

According to the special theory of relativity, this value is low by a factor of 2. Thus, in vector form, the spin magnetic moment is

$$\vec{\boldsymbol{\mu}} = \left(\frac{e}{m_e}\right)\vec{\mathbf{S}}. \tag{8.31}$$

The *z*-component of the magnetic moment is

$$\mu_z = -\left(\frac{e}{m_e}\right)S_z = -\left(\frac{e}{m_e}\right)m_s\hbar. \tag{8.32}$$

The spin projection quantum number has just two values $(m_s = \pm 1/2)$, so the *z*-component of the magnetic moment also has just two values:

$$\mu_z = \pm\left(\frac{e}{2m_e}\right) = \pm\mu_B\hbar, \tag{8.33}$$

where μ_B is one Bohr magneton. An electron is magnetic, so we expect the electron to interact with other magnetic fields. We consider two special cases: the interaction of a free electron with an external (nonuniform) magnetic field, and an electron in a hydrogen atom with a magnetic field produced by the orbital angular momentum of the electron.

Example 8.4

Electron Spin and Radiation

A hydrogen atom in the ground state is placed in an external uniform magnetic field ($B = 1.5\text{ T}$). Determine the frequency of radiation produced in a transition between the spin-up and spin-down states of the electron.

Strategy

The spin projection quantum number is $m_s = \pm 1/2$, so the *z*-component of the magnetic moment is

$$\mu_z = \pm\left(\frac{e}{2m_e}\right) = \pm\mu_B \hbar.$$

The potential energy associated with the interaction between the electron magnetic moment and the external magnetic field is

$$U = -\mu_z B = \mp\mu_B B.$$

The frequency of light emitted is proportional to the energy (ΔE) difference between these two states.

Solution

The energy difference between these states is $\Delta E = 2\mu_B B$, so the frequency of radiation produced is

$$f = \frac{\Delta E}{h} = \frac{2\mu_B B}{h} = \frac{2\left(5.79 \times \frac{10^{-5}\ \text{eV}}{\text{T}}\right)(1.5\text{T})}{4.136 \times 10^{-15}\ \text{eV} \cdot \text{s}} = 4.2 \times 10^{10} \frac{\text{cycles}}{\text{s}}.$$

Significance

The electron magnetic moment couples with the external magnetic field. The energy of this system is different whether the electron is aligned or not with the proton. The frequency of radiation produced by a transition between these states is proportional to the energy difference. If we double the strength of the magnetic field, holding all other things constant, the frequency of the radiation doubles and its wavelength is cut in half.

In a hydrogen atom, the electron magnetic moment can interact with the magnetic field produced by the orbital angular momentum of the electron, a phenomenon called **spin-orbit coupling**. The orbital angular momentum ($\vec{L}$), orbital magnetic moment ($\vec{\mu}$), spin angular momentum ($\vec{S}$), and spin magnetic moment ($\vec{\mu}_s$) vectors are shown together in **Figure 8.14**.

Just as the energy levels of a hydrogen atom can be split by an *external* magnetic field, so too are the energy levels of a hydrogen atom split by *internal* magnetic fields of the atom. If the magnetic moment of the electron and orbital magnetic moment of the electron are antiparallel, the potential energy from the magnetic interaction is relatively high, but when these moments are parallel, the potential energy is relatively small. Transition from each of these two states to a lower-energy level results in the emission of a photon of slightly different frequency. That is, the spin-orbit coupling "splits" the spectral line expected from a spin-less electron. The **fine structure** of the hydrogen spectrum is explained by spin-orbit coupling.

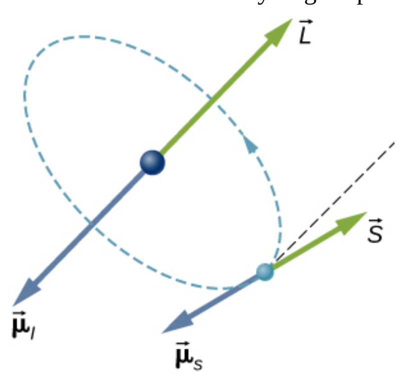


Figure 8.14 Spin-orbit coupling is the interaction of an electron's spin magnetic moment $\vec{\mu}_s$ with its orbital magnetic moment $\vec{\mu}_l$.

The Stern-Gerlach experiment provides experimental evidence that electrons have spin angular momentum. The experiment passes a stream of silver (Ag) atoms through an external, nonuniform magnetic field. The Ag atom has an orbital angular momentum of zero and contains a single unpaired electron in the outer shell. Therefore, the total angular momentum of the Ag atom is due entirely to the spin of the outer electron ($s = 1/2$) . Due to electron spin, the Ag atoms act as tiny magnets as they pass through the magnetic field. These “magnets” have two possible orientations, which correspond to the spin-up and -down states of the electron. The magnetic field diverts the spin up atoms in one direction and the spin-down atoms in another direction. This produces two distinct bands on a screen (**Figure 8.15**).

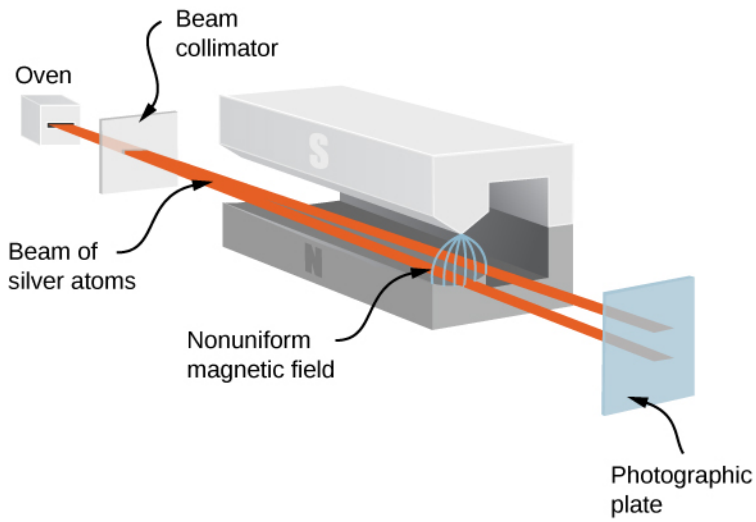


Figure 8.15 In the Stern-Gerlach experiment, an external, nonuniform magnetic field diverts a beam of electrons in two different directions. This result is due to the quantization of spin angular momentum.

According to classical predictions, the angular momentum (and, therefore, the magnetic moment) of the Ag atom can point in any direction, so one expects, instead, a continuous smudge on the screen. The resulting two bands of the Stern-Gerlach experiment provide startling support for the ideas of quantum mechanics.

Visit **PhET Explorations: Stern-Gerlach Experiment (https://openstaxcollege.org/l/21sterngerlach)** to learn more about the Stern-Gerlach experiment.

8.2 Check Your Understanding If the Stern-Gerlach experiment yielded four distinct bands instead of two, what might be concluded about the spin quantum number of the charged particle?

Just like an electron, a proton is spin 1/2 and has a magnetic moment. (According to nuclear theory, this moment is due to the orbital motion of quarks within the proton.) The **hyperfine structure** of the hydrogen spectrum is explained by the interaction between the magnetic moment of the proton and the magnetic moment of the electron, an interaction known as spin-spin coupling. The energy of the electron-proton system is different depending on whether or not the moments are aligned. Transitions between these states (**spin-flip transitions**) result in the emission of a photon with a wavelength of $\lambda \approx 21\text{ cm}$ (in the radio range). The 21-cm line in atomic spectroscopy is a “fingerprint” of hydrogen gas. Astronomers exploit this spectral line to map the spiral arms of galaxies, which are composed mostly of hydrogen (**Figure 8.16**).

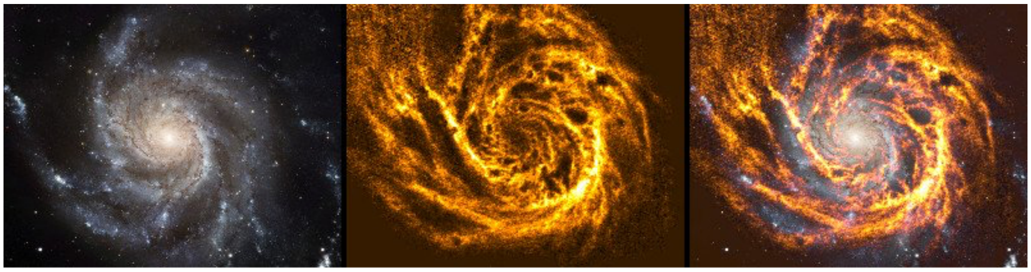


Figure 8.16 The magnetic interaction between the electron and proton in the hydrogen atom is used to map the spiral arms of the Pinwheel Galaxy (NGC 5457). (a) The galaxy seen in visible light; (b) the galaxy seen in 21-cm hydrogen radiation; (c) the composite image of (a) and (b). Notice how the hydrogen emission penetrates dust in the galaxy to show the spiral arms very clearly, whereas the galactic nucleus shows up better in visible light (credit a: modification of work by ESA & NASA; credit b: modification of work by Fabian Walter).

A complete specification of the state of an electron in a hydrogen atom requires five quantum numbers: *n*, *l*, *m*, *s*, and m_s. The names, symbols, and allowed values of these quantum numbers are summarized in **Table 8.4**.

Name	Symbol	Allowed values
Principal quantum number	*n*	1, 2, 3, …
Angular momentum	*l*	0, 1, 2, … *n* – 1
Angular momentum projection	*m*	$0, \ \pm 1, \ \pm 2, ... \ \pm l$
Spin	*s*	1/2 (electrons)
Spin projection	m_s	$-½, \ +½$

Table 8.4 Summary of Quantum Numbers of an Electron in a Hydrogen Atom

Note that the intrinsic quantum numbers introduced in this section (*s* and m_s) are valid for many particles, not just electrons. For example, quarks within an atomic nucleus are also spin-half particles. As we will see later, quantum numbers help to classify subatomic particles and enter into scientific models that attempt to explain how the universe works.

8.4 | The Exclusion Principle and the Periodic Table

Learning Objectives

By the end of this section, you will be able to:

- Explain the importance of Pauli's exclusion principle to an understanding of atomic structure and molecular bonding
- Explain the structure of the periodic table in terms of the total energy, orbital angular momentum, and spin of individual electrons in an atom
- Describe the electron configuration of atoms in the periodic table

So far, we have studied only hydrogen, the simplest chemical element. We have found that an electron in the hydrogen atom can be completely specified by five quantum numbers:

n : principal quantum number **(8.34)**
l : angular momentum quantum number
m : angular momentum projection quantum number
s : spin quantum number
m_s : spin projection quantum number

To construct the ground state of a neutral multi-electron atom, imagine starting with a nucleus of charge Ze (that is, a nucleus of atomic number Z) and then adding Z electrons one by one. Assume that each electron moves in a spherically symmetrical electric field produced by the nucleus and all other electrons of the atom. The assumption is valid because the electrons are distributed randomly around the nucleus and produce an average electric field (and potential) that is spherically symmetrical. The electric potential $U(r)$ for each electron does not follow the simple $-1/r$ form because of interactions between electrons, but it turns out that we can still label each individual electron state by quantum numbers, (n, l, m, s, m_s). (The spin quantum number s is the same for all electrons, so it will not be used in this section.)

The structure and chemical properties of atoms are explained in part by **Pauli's exclusion principle**: No two electrons in an atom can have the same values for all four quantum numbers (n, l, m, m_s). This principle is related to two properties of electrons: All electrons are identical ("when you've seen one electron, you've seen them all") and they have half-integral spin $(s = 1/2)$. Sample sets of quantum numbers for the electrons in an atom are given in **Table 8.5**. Consistent with Pauli's exclusion principle, no two rows of the table have the exact same set of quantum numbers.

n	**l**	**m**	m_s	**Subshell symbol**	**No. of electrons: subshell**	**No. of electrons: shell**
1	0	0	½	1*s*	2	2
1	0	0	–½			
2	0	0	½	2*s*	2	8
2	0	0	–½			
2	1	–1	½	2*p*	6	
2	1	–1	–½			
2	1	0	½			
2	1	0	–½			
2	1	1	½			
2	1	1	–½			
3	0	0	½	3*s*	2	18
3	0	0	–½			
3	1	–1	½	3*p*	6	
3	1	–1	–½			
3	1	0	½			
3	1	0	–½			
3	1	1	½			
3	1	1	–½			
3	2	–2	½	3*d*	10	
3	2	–2	–½			
3	2	–1	½			

Table 8.5 Electron States of Atoms Because of Pauli's exclusion principle, no two electrons in an atom have the same set of four quantum numbers.

n	*l*	*m*	m_s	Subshell symbol	No. of electrons: subshell	No. of electrons: shell
3	2	–1	–½			
3	2	0	½			
3	2	0	–½			
3	2	1	½			
3	2	1	–½			
3	2	2	½			
3	2	2	–½			

Table 8.5 Electron States of Atoms Because of Pauli's exclusion principle, no two electrons in an atom have the same set of four quantum numbers.

Electrons with the same principal quantum number *n* are said to be in the same shell, and those that have the same value of *l* are said to occupy the same subshell. An electron in the $n = 1$ state of a hydrogen atom is denoted 1*s*, where the first digit indicates the shell $(n = 1)$ and the letter indicates the subshell $(s, p, d, f\ldots$ correspond to $l = 0, 1, 2, 3\ldots)$. Two electrons in the $n = 1$ state are denoted as $1s^2$, where the superscript indicates the number of electrons. An electron in the $n = 2$ state with $l = 1$ is denoted 2*p*. The combination of two electrons in the $n = 2$ and $l = 0$ state, and three electrons in the $n = 2$ and $l = 1$ state is written as $2s^2 2p^3$, and so on. This representation of the electron state is called the **electron configuration** of the atom. The electron configurations for several atoms are given in **Table 8.6**. Electrons in the outer shell of an atom are called **valence electrons**. Chemical bonding between atoms in a molecule are explained by the transfer and sharing of valence electrons.

Element	Electron Configuration	Spin Alignment
H	$1s^1$	$(\uparrow)$
He	$1s^2$	$(\uparrow\downarrow)$
Li	$1s^2 2s^1$	$(\uparrow)$
Be	$1s^2 2s^2$	$(\uparrow\downarrow)$
B	$1s^2 2s^2 2p^1$	$(\uparrow\downarrow)(\uparrow)$
C	$1s^2 2s^2 2p^2$	$(\uparrow\downarrow)(\uparrow)(\uparrow)$
N	$1s^2 2s^2 2p^3$	$(\uparrow\downarrow)(\uparrow)(\uparrow)(\uparrow)$
O	$1s^2 2s^2 2p^4$	$(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow)(\uparrow)$
F	$1s^2 2s^2 2p^5$	$(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow)$
Ne	$1s^2 2s^2 2p^6$	$(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)(\uparrow\downarrow)$
Na	$1s^2 2s^2 2p^6 3s^1$	$(\uparrow)$

Table 8.6 Electron Configurations of Electrons in an Atom The symbol $(\uparrow)$ indicates an unpaired electron in the outer shell, whereas the symbol $(\uparrow\downarrow)$ indicates a pair of spin-up and -down electrons in an outer shell.

Element	Electron Configuration	Spin Alignment
Mg	$1s^2 2s^2 2p^6 3s^2$	$(\uparrow\downarrow)$
Al	$1s^2 2s^2 2p^6 3s^2 3p^1$	$(\uparrow\downarrow)(\uparrow)$

Table 8.6 Electron Configurations of Electrons in an Atom The symbol $(\uparrow)$ indicates an unpaired electron in the outer shell, whereas the symbol $(\uparrow\downarrow)$ indicates a pair of spin-up and -down electrons in an outer shell.

The maximum number of electrons in a subshell depends on the value of the angular momentum quantum number, *l*. For a given a value *l*, there are $2l + 1$ orbital angular momentum states. However, each of these states can be filled by two electrons (spin up and down, $\uparrow\downarrow$). Thus, the maximum number of electrons in a subshell is

$$N = 2(2l + 1) = 4l + 2. \tag{8.35}$$

In the 2*s* $(l = 0)$ subshell, the maximum number of electrons is 2. In the 2*p* ($l = 1$) subshell, the maximum number of electrons is 6. Therefore, the total maximum number of electrons in the $n = 2$ shell (including both the $l = 0$ and 1 subshells) is $2 + 6$ or 8. In general, the maximum number of electrons in the *n*th shell is $2n^2$.

Example 8.5

Subshells and Totals for $n = 3$

How many subshells are in the $n = 3$ shell? Identify each subshell and calculate the maximum number of electrons that will fill each. Show that the maximum number of electrons that fill an atom is $2n^2$.

Strategy

Subshells are determined by the value of *l*; thus, we first determine which values of *l* are allowed, and then we apply the equation "maximum number of electrons that can be in a subshell $= 2(2l + 1)$" to find the number of electrons in each subshell.

Solution

Because $n = 3$, we know that *l* can be 0, 1, or 2; thus, there are three possible subshells. In standard notation, they are labeled the 3*s*, 3*p*, and 3*d* subshells. We have already seen that two electrons can be in an *s* state, and six in a *p* state, but let us use the equation "maximum number of electrons that can be in a subshell $= 2(2l + 1)$" to calculate the maximum number in each:

$$\begin{aligned}
&3s \text{ has } l = 0;\ \text{thus, } 2(2l + 1) = 2(0 + 1) = 2 \\
&3p \text{ has } l = 1;\ \text{thus, } 2(2l + 1) = 2(2 + 1) = 6 \\
&3d \text{ has } l = 2;\ \text{thus, } 2(2l + 1) = 2(4 + 1) = 10 \\
&\text{Total} = 18 \\
&(\text{in the } n = 3 \text{ shell}).
\end{aligned}$$

The equation "maximum number of electrons that can be in a shell $= 2n^2$" gives the maximum number in the $n = 3$ shell to be

$$\text{Maximum number of electrons} = 2n^2 = 2(3)^2 = 2(9) = 18.$$

Significance

The total number of electrons in the three possible subshells is thus the same as the formula $2n^2$. In standard (spectroscopic) notation, a filled $n = 3$ shell is denoted as $3s^2 3p^6 3d^{10}$. Shells do not fill in a simple manner. Before the $n = 3$ shell is completely filled, for example, we begin to find electrons in the $n = 4$ shell.

The structure of the periodic table (**Figure 8.17**) can be understood in terms of shells and subshells, and, ultimately, the total energy, orbital angular momentum, and spin of the electrons in the atom. A detailed discussion of the periodic table is left to a chemistry course—we sketch only its basic features here. In this discussion, we assume that the atoms are electrically neutral; that is, they have the same number of electrons and protons. (Recall that the total number of protons in an atomic nucleus is called the atomic number, *Z*.)

First, the periodic table is arranged into columns and rows. The table is read left to right and top to bottom in the order of increasing atomic number *Z*. Atoms that belong to the same column or **chemical group** share many of the same chemical properties. For example, the Li and Na atoms (in the first column) bond to other atoms in a similar way. The first row of the table corresponds to the 1*s* ($l = 0$) shell of an atom.

Consider the hypothetical procedure of adding electrons, one by one, to an atom. For hydrogen (H) (upper left), the 1*s* shell is filled with either a spin up or down electron ($\uparrow$ or $\downarrow$). This lone electron is easily shared with other atoms, so hydrogen is chemically active. For helium (He) (upper right), the 1*s* shell is filled with both a spin up and a spin down ($\uparrow\downarrow$) electron. This “fills” the 1*s* shell, so a helium atom tends not to share electrons with other atoms. The helium atom is said to be chemically inactive, inert, or noble; likewise, helium gas is said to be an inert gas or noble gas.

Build an atom by adding and subtracting protons, neutrons, and electrons. How does the element, charge, and mass change? Visit **PhET Explorations: Build an Atom (https://openstaxcollege.org/l/21buildanatom)** to explore the answers to these questions.

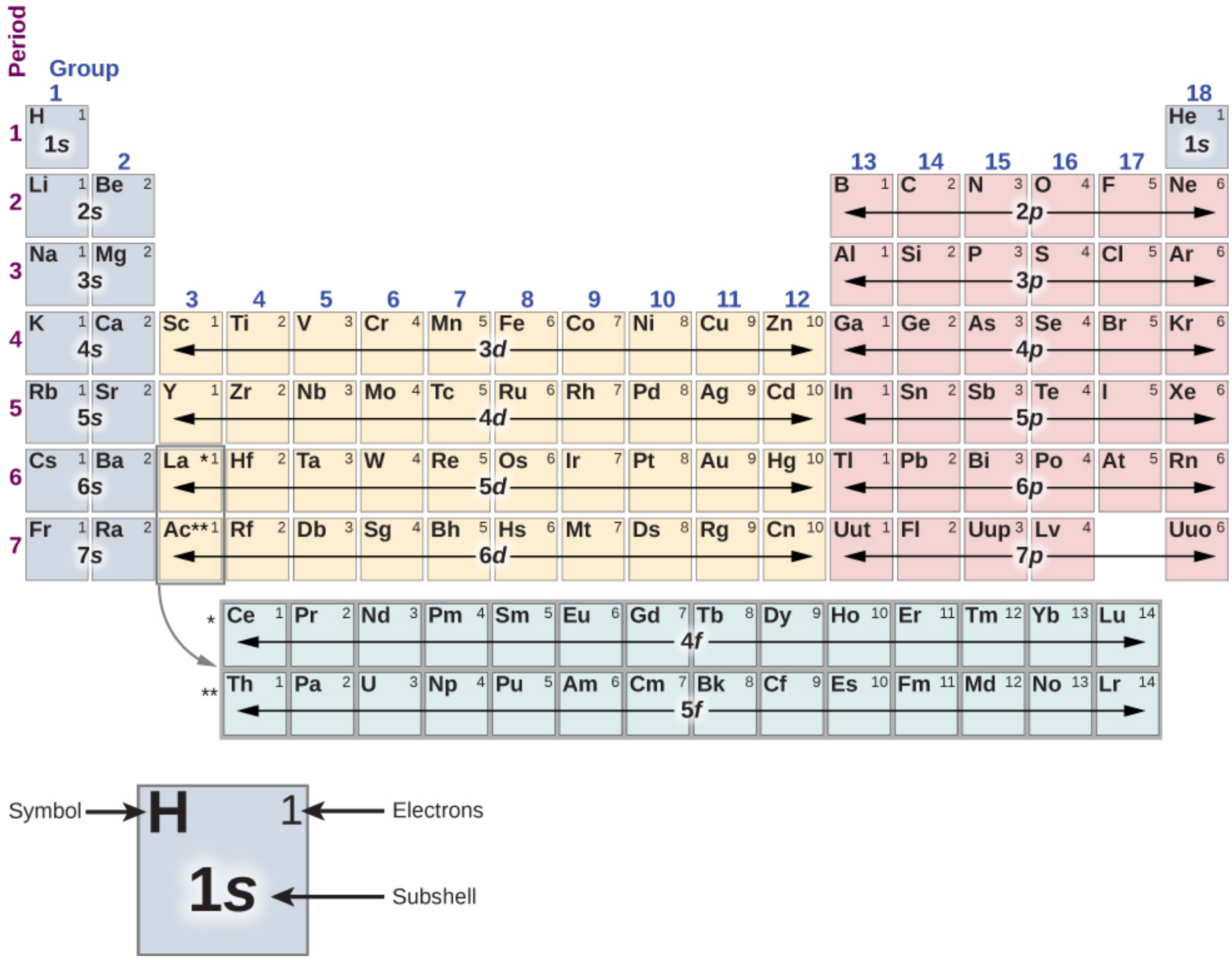


Figure 8.17 The periodic table of elements, showing the structure of shells and subshells.

The second row corresponds to the 2*s* and 2*p* subshells. For lithium (Li) (upper left), the 1*s* shell is filled with a spin-up *and* spin-down electron ($\uparrow\downarrow$) and the 2*s* shell is filled with either a spin-up or -down electron ($\uparrow$or$\downarrow$). Its electron configuration is therefore $1s^2 2s^1$ or [He]2*s*, where [He] indicates a helium core. Like hydrogen, the lone electron in the outermost shell is easily shared with other atoms. For beryllium (Be), the 2*s* shell is filled with a spin-up and -down electron ($\uparrow\downarrow$), and has the electron configuration [He] $2s^2$.

Next, we look at the right side of the table. For boron (B), the 1*s* and 2*s* shells are filled and the 2*p* ($l = 1$) shell contains either a spin up or down electron ($\uparrow$or$\downarrow$). From carbon (C) to neon (N), we the fill the 2*p* shell. The maximum number of electrons in the 2*p* shells is $4l + 2 = 4(2) + 2 = 6$. For neon (Ne), the 1*s* shell is filled with a spin-up and spin-down electron ($\uparrow\downarrow$), and the 2*p* shell is filled with six electrons ($\uparrow\downarrow\ \uparrow\downarrow\ \uparrow\downarrow$) . This “fills” the 1*s*, 2*s*, and 2*p* subshells, so like helium, the neon atom tends not to share electrons with other atoms.

The process of electron filling repeats in the third row. However, beginning in the fourth row, the pattern is broken. The actual order of order of electron filling is given by

1*s*, 2*s*, 2*p*, 3*s*, 3*p*, 4*s*, **3*d***, 4*p*, 5*s*, **4*d***, 5*p*, 6*s*, **4*f*, 5*d***, 6*p*, 7*s*,...

Notice that the 3*d*, 4*d*, 4*f*, and 5*d* subshells (in bold) are filled out of order; this occurs because of interactions between electrons in the atom, which so far we have neglected. The **transition metals** are elements in the gap between the first two columns and the last six columns that contain electrons that fill the *d* ($l = 1$) subshell. As expected, these atoms are arranged in $4l + 2 = 4(2) + 2 = 10$ columns. The structure of the periodic table can be understood in terms of the quantization of the total energy (*n*), orbital angular momentum (*l*), and spin (*s*). The first two columns correspond to the s ($l = 0$) subshell, the next six columns correspond to the p ($l = 1$) subshell, and the gap between these columns corresponds to the d ($l = 2$) subshell.

The periodic table also gives information on molecular bonding. To see this, consider atoms in the left-most column (the so-called alkali metals including: Li, Na, and K). These atoms contain a single electron in the 2*s* subshell, which is easily donated to other atoms. In contrast, atoms in the second-to-right column (the halogens: for example, Cl, F, and Br) are relatively stingy in sharing electrons. These atoms would much rather accept an electron, because they are just one electron shy of a filled shell ("of being noble").

Therefore, if a Na atom is placed in close proximity to a Cl atom, the Na atom freely donates its 2*s* electron and the Cl atom eagerly accepts it. In the process, the Na atom (originally a neutral charge) becomes positively charged and the Cl (originally a neutral charge) becomes negatively charged. Charged atoms are called ions. In this case, the ions are Na^+ and Cl^-, where the superscript indicates charge of the ion. The electric (Coulomb) attraction between these atoms forms a NaCl (salt) molecule. A chemical bond between two ions is called an **ionic bond**. There are many kinds of chemical bonds. For example, in an oxygen molecule O_2 electrons are equally shared between the atoms. The bonding of oxygen atoms is an example of a **covalent bond**.

8.5 | Atomic Spectra and X-rays

Learning Objectives

By the end of this section, you will be able to:

- Describe the absorption and emission of radiation in terms of atomic energy levels and energy differences
- Use quantum numbers to estimate the energy, frequency, and wavelength of photons produced by atomic transitions in multi-electron atoms
- Explain radiation concepts in the context of atomic fluorescence and X-rays

The study of atomic spectra provides most of our knowledge about atoms. In modern science, atomic spectra are used to identify species of atoms in a range of objects, from distant galaxies to blood samples at a crime scene.

The theoretical basis of atomic spectroscopy is the transition of electrons between energy levels in atoms. For example, if an electron in a hydrogen atom makes a transition from the $n = 3$ to the $n = 2$ shell, the atom emits a photon with a wavelength

$$\lambda = \frac{c}{f} = \frac{h \cdot c}{h \cdot f} = \frac{hc}{\Delta E} = \frac{hc}{E_3 - E_2}, \quad (8.36)$$

where $\Delta E = E_3 - E_2$ is energy carried away by the photon and $hc = 1940\ \text{eV} \cdot \text{nm}$. After this radiation passes through a spectrometer, it appears as a sharp spectral line on a screen. The Bohr model of this process is shown in **Figure 8.18**. If the electron later absorbs a photon with energy ΔE, the electron returns to the $n = 3$ shell. (We examined the Bohr model earlier, in **Photons and Matter Waves**.)

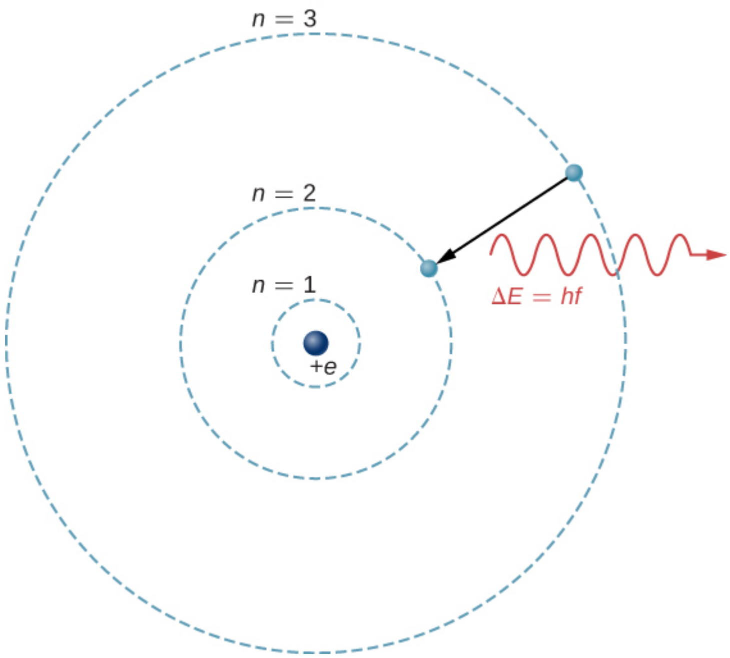


Figure 8.18 An electron transition from the $n = 3$ to the $n = 2$ shell of a hydrogen atom.

To understand atomic transitions in multi-electron atoms, it is necessary to consider many effects, including the Coulomb repulsion between electrons and internal magnetic interactions (spin-orbit and spin-spin couplings). Fortunately, many properties of these systems can be understood by neglecting interactions between electrons and representing each electron by its own single-particle wave function ψ_{nlm}.

Atomic transitions must obey **selection rules**. These rules follow from principles of quantum mechanics and symmetry. Selection rules classify transitions as either allowed or forbidden. (Forbidden transitions do occur, but the probability of the typical forbidden transition is very small.) For a hydrogen-like atom, atomic transitions that involve electromagnetic interactions (the emission and absorption of photons) obey the following selection rule:

$$\Delta l = \pm 1, \quad (8.37)$$

where l is associated with the magnitude of orbital angular momentum,

$$L = \sqrt{l(l+1)}\hbar. \quad (8.38)$$

For multi-electron atoms, similar rules apply. To illustrate this rule, consider the observed atomic transitions in hydrogen (H), sodium (Na), and mercury (Hg) (**Figure 8.19**). The horizontal lines in this diagram correspond to atomic energy levels, and the transitions allowed by this selection rule are shown by lines drawn between these levels. The energies of these states are on the order of a few electron volts, and photons emitted in transitions are in the visible range. Technically, atomic transitions can violate the selection rule, but such transitions are uncommon.

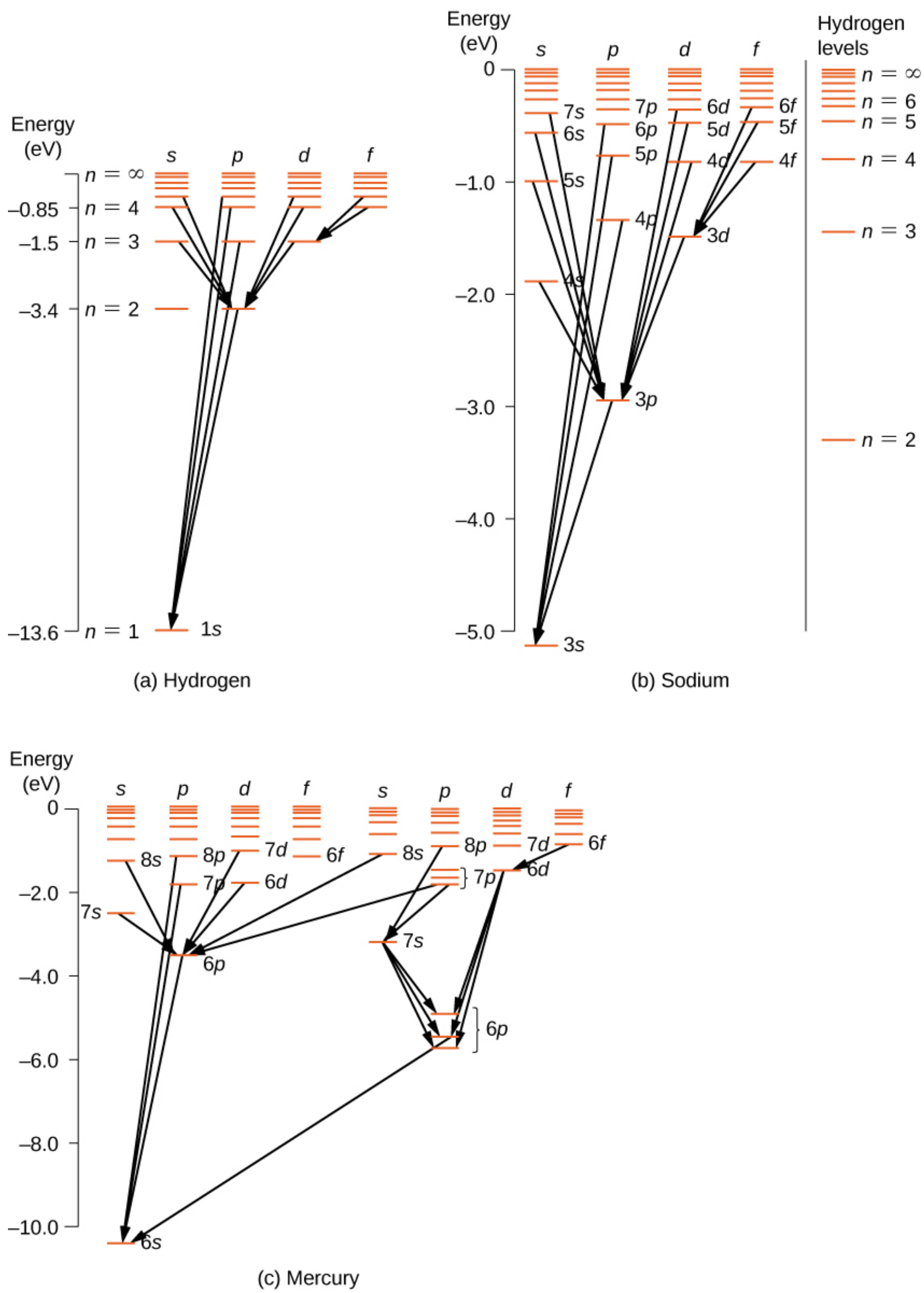


Figure 8.19 Energy-level diagrams for (a) hydrogen, (b) sodium, and (c) mercury. For comparison, hydrogen energy levels are shown in the sodium diagram.

The hydrogen atom has the simplest energy-level diagram. If we neglect electron spin, all states with the same value of n have the same total energy. However, spin-orbit coupling splits the $n = 2$ states into two angular momentum states (*s* and *p*) of slightly different energies. (These levels are not vertically displaced, because the energy splitting is too small to show up in this diagram.) Likewise, spin-orbit coupling splits the $n = 3$ states into three angular momentum states (*s*, *p*, and *d*).

The energy-level diagram for hydrogen is similar to sodium, because both atoms have one electron in the outer shell. The valence electron of sodium moves in the electric field of a nucleus shielded by electrons in the inner shells, so it does not experience a simple $1/r$ Coulomb potential and its total energy depends on both n and l. Interestingly, mercury has two separate energy-level diagrams; these diagrams correspond to two net spin states of its 6*s* (valence) electrons.

Example 8.6

The Sodium Doublet

The spectrum of sodium is analyzed with a spectrometer. Two closely spaced lines with wavelengths 589.00 nm and 589.59 nm are observed. (a) If the doublet corresponds to the excited (valence) electron that transitions from some excited state down to the 3*s* state, what was the original electron angular momentum? (b) What is the energy difference between these two excited states?

Strategy

Sodium and hydrogen belong to the same column or chemical group of the periodic table, so sodium is "hydrogen-like." The outermost electron in sodium is in the 3*s* ($l = 0$) subshell and can be excited to higher energy levels. As for hydrogen, subsequent transitions to lower energy levels must obey the selection rule:

$$\Delta l = \pm 1.$$

We must first determine the quantum number of the initial state that satisfies the selection rule. Then, we can use this number to determine the magnitude of orbital angular momentum of the initial state.

Solution

a. Allowed transitions must obey the selection rule. If the quantum number of the initial state is $l = 0$, the transition is forbidden because $\Delta l = 0$. If the quantum number of the initial state is $l = 2, 3, 4, \ldots$ the transition is forbidden because $\Delta l > 1$. Therefore, the quantum of the initial state must be $l = 1$. The orbital angular momentum of the initial state is

$$L = \sqrt{l(l+1)}\hbar = 1.41\hbar.$$

b. Because the final state for both transitions is the same (3*s*), the difference in energies of the photons is equal to the difference in energies of the two excited states. Using the equation

$$\Delta E = hf = h\left(\frac{c}{\lambda}\right),$$

we have

$$\begin{aligned}\Delta E &= hc\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) \\ &= \left(4.14 \times 10^{-15}\ \text{eVs}\right)\left(3.00 \times 10^{8}\ \text{m/s}\right) \times \left(\frac{1}{589.00 \times 10^{-9}\ \text{m}} - \frac{1}{589.59 \times 10^{-9}\ \text{m}}\right) \\ &= 2.11 \times 10^{-3}\ \text{eV}.\end{aligned}$$

Significance

To understand the difficulty of measuring this energy difference, we compare this difference with the average energy of the two photons emitted in the transition. Given an average wavelength of 589.30 nm, the average energy of the photons is

$$E = \frac{hc}{\lambda} = \frac{(4.14 \times 10^{-15}\ \text{eVs})(3.00 \times 10^{8}\ \text{m/s})}{589.30 \times 10^{-9}\ \text{m}} = 2.11\ \text{eV}.$$

The energy difference ΔE is about 0.1% (1 part in 1000) of this average energy. However, a sensitive spectrometer can measure the difference.

Atomic Fluorescence

Fluorescence occurs when an electron in an atom is excited several steps above the ground state by the absorption of a high-energy ultraviolet (UV) photon. Once excited, the electron "de-excites" in two ways. The electron can drop back to the ground state, emitting a photon of the same energy that excited it, or it can drop in a series of smaller steps, emitting several low-energy photons. Some of these photons may be in the visible range. Fluorescent dye in clothes can make colors seem brighter in sunlight by converting UV radiation into visible light. Fluorescent lights are more efficient in converting electrical energy into visible light than incandescent filaments (about four times as efficient). **Figure 8.20** shows a scorpion illuminated by a UV lamp. Proteins near the surface of the skin emit a characteristic blue light.

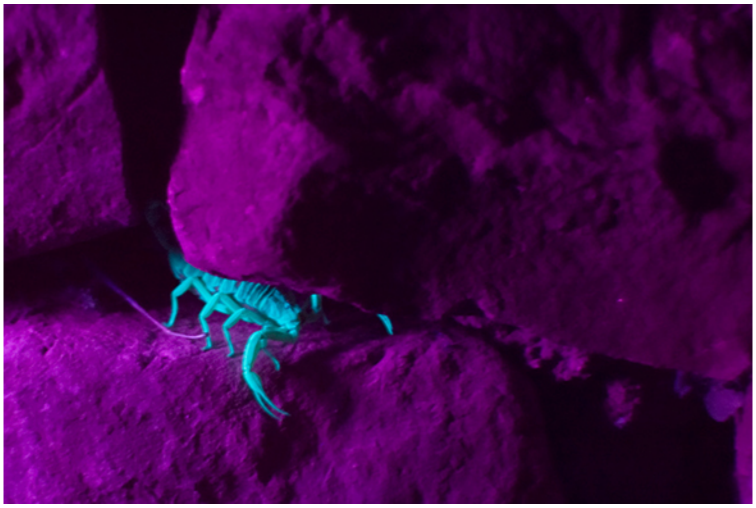


Figure 8.20 A scorpion glows blue under a UV lamp. (credit: Ken Bosma)

X-rays

The study of atomic energy transitions enables us to understand X-rays and X-ray technology. Like all electromagnetic radiation, X-rays are made of photons. X-ray photons are produced when electrons in the outermost shells of an atom drop to the inner shells. (Hydrogen atoms do not emit X-rays, because the electron energy levels are too closely spaced together to permit the emission of high-frequency radiation.) Transitions of this kind are normally forbidden because the lower states are already filled. However, if an inner shell has a vacancy (an inner electron is missing, perhaps from being knocked away by a high-speed electron), an electron from one of the outer shells can drop in energy to fill the vacancy. The energy gap for such a transition is relatively large, so wavelength of the radiated X-ray photon is relatively short.

X-rays can also be produced by bombarding a metal target with high-energy electrons, as shown in **Figure 8.21**. In the figure, electrons are boiled off a filament and accelerated by an electric field into a tungsten target. According to the classical theory of electromagnetism, *any* charged particle that accelerates emits radiation. Thus, when the electron strikes the tungsten target, and suddenly slows down, the electron emits **braking radiation**. (Braking radiation refers to radiation produced by any charged particle that is slowed by a medium.) In this case, braking radiation contains a continuous range of frequencies, because the electrons will collide with the target atoms in slightly different ways.

Braking radiation is not the only type of radiation produced in this interaction. In some cases, an electron collides with another inner-shell electron of a target atom, and knocks the electron out of the atom—billiard ball style. The empty state is filled when an electron in a higher shell drops into the state (drop in energy level) and emits an X-ray photon.

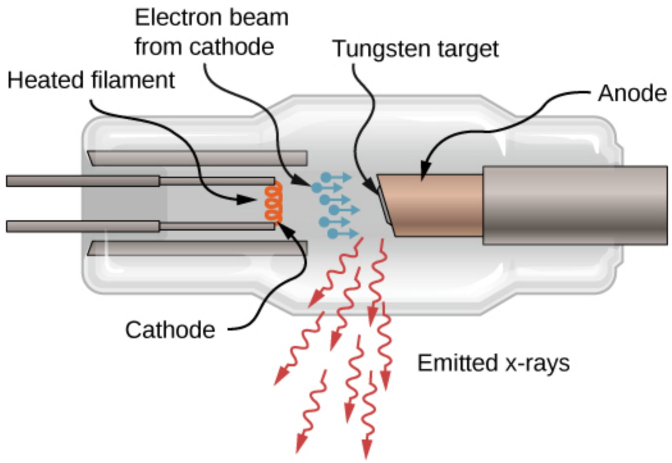


Figure 8.21 A sketch of an X-ray tube. X-rays are emitted from the tungsten target.

Historically, X-ray spectral lines were labeled with letters (*K*, *L*, *M*, *N*, …). These letters correspond to the atomic shells ($n = 1, 2, 3, 4, \ldots$). X-rays produced by a transition from any higher shell to the *K* ($n = 1$) shell are labeled as *K* X-rays. X-rays produced in a transition from the *L* ($n = 2$) shell are called K_α X-rays; X-rays produced in a transition from the *M* ($n = 3$) shell are called K_β X-rays; X-rays produced in a transition from the *N* ($n = 4$) shell are called K_γ X-rays; and so forth. Transitions from higher shells to *L* and *M* shells are labeled similarly. These transitions are represented by an energy-level diagram in **Figure 8.22**.

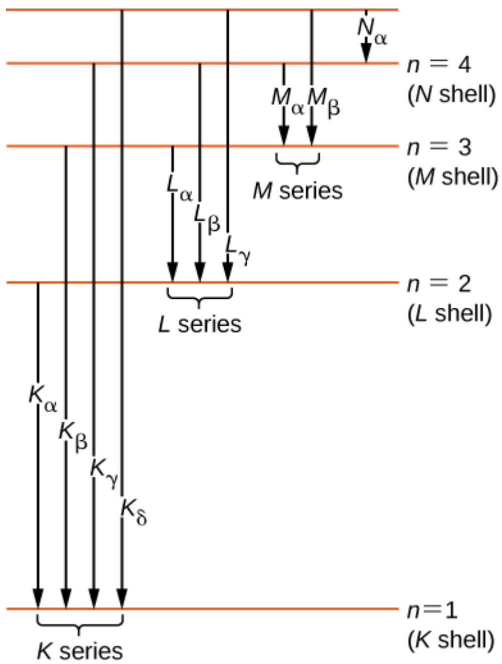


Figure 8.22 X-ray transitions in an atom.

The distribution of X-ray wavelengths produced by striking metal with a beam of electrons is given in **Figure 8.23**. X-ray transitions in the target metal appear as peaks on top of the braking radiation curve. Photon frequencies corresponding to the spikes in the X-ray distribution are called characteristic frequencies, because they can be used to identify the target metal. The sharp cutoff wavelength (just below the K_γ peak) corresponds to an electron that loses all of its energy to a single photon. Radiation of shorter wavelengths is forbidden by the conservation of energy.

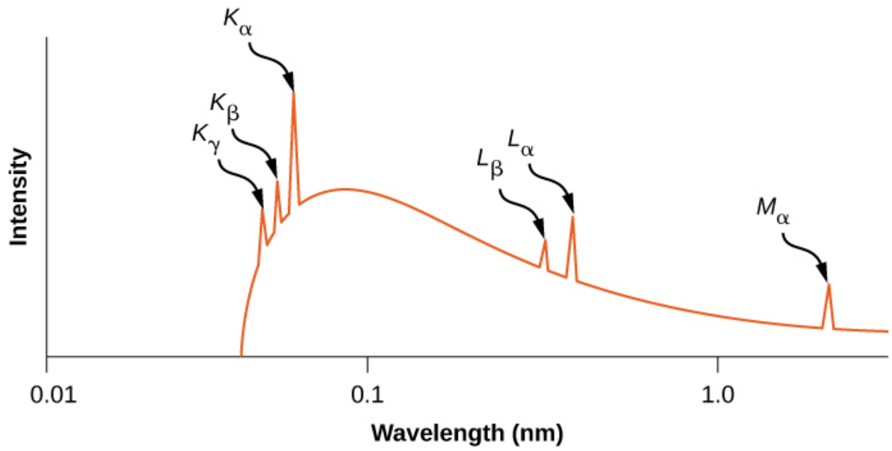


Figure 8.23 X-ray spectrum from a silver target. The peaks correspond to characteristic frequencies of X-rays emitted by silver when struck by an electron beam.

Example 8.7

X-Rays from Aluminum

Estimate the characteristic energy and frequency of the K_α X-ray for aluminum ($Z = 13$).

Strategy

A K_α X-ray is produced by the transition of an electron in the L ($n = 2$) shell to the K ($n = 1$) shell. An electron in the L shell "sees" a charge $Z = 13 - 1 = 12,$ because one electron in the K shell shields the nuclear charge. (Recall, two electrons are not in the K shell because the other electron state is vacant.) The frequency of the emitted photon can be estimated from the energy difference between the L and K shells.

Solution

The energy difference between the L and K shells in a hydrogen atom is 10.2 eV. Assuming that other electrons in the L shell or in higher-energy shells do not shield the nuclear charge, the energy difference between the L and K shells in an atom with $Z = 13$ is approximately

$$\Delta E_{L \to K} \approx (Z-1)^2(10.2\text{ eV}) = (13-1)^2(10.2\text{ eV}) = 1.47 \times 10^3\text{ eV}. \quad (8.39)$$

Based on the relationship $f = (\Delta E_{L \to K})/h$, the frequency of the X-ray is

$$f = \frac{1.47 \times 10^3\text{ eV}}{4.14 \times 10^{-15}\text{ eV} \cdot \text{s}} = 3.55 \times 10^{17}\text{ Hz}.$$

Significance

The wavelength of the typical X-ray is 0.1–10 nm. In this case, the wavelength is:

$$\lambda = \frac{c}{f} = \frac{3.0 \times 10^8\text{ m/s}}{3.55 \times 10^{17}\text{ Hz}} = 8.5 \times 10^{-10} = 0.85\text{ nm}.$$

Hence, the transition $L \to K$ in aluminum produces X-ray radiation.

X-ray production provides an important test of quantum mechanics. According to the Bohr model, the energy of a K_α X-ray depends on the nuclear charge or atomic number, Z. If Z is large, Coulomb forces in the atom are large, energy differences (ΔE) are large, and, therefore, the energy of radiated photons is large. To illustrate, consider a single electron in a multi-electron atom. Neglecting interactions between the electrons, the allowed energy levels are

$$E_n = -\frac{Z^2(13.6\text{ eV})}{n^2}, \quad (8.40)$$

where n = 1, 2, …and Z is the atomic number of the nucleus. However, an electron in the L ($n = 2$) shell "sees" a charge $Z - 1$, because one electron in the K shell shields the nuclear charge. (Recall that there is only one electron in the K shell because the other electron was "knocked out.") Therefore, the approximate energies of the electron in the L and K shells are

$$E_L \approx -\frac{(Z-1)^2(13.6\text{ eV})}{2^2}$$
$$E_K \approx -\frac{(Z-1)^2(13.6\text{ eV})}{1^2}.$$

The energy carried away by a photon in a transition from the L shell to the K shell is therefore

$$\begin{aligned}\Delta E_{L\to K} &= (Z-1)^2(13.6\text{ eV})\left(\frac{1}{1^2}-\frac{1}{2^2}\right)\\ &= (Z-1)^2(10.2\text{ eV}),\end{aligned}$$

where Z is the atomic number. In general, the X-ray photon energy for a transition from an outer shell to the K shell is

$$\Delta E_{L\to K} = hf = \text{constant} \times (Z-1)^2,$$

or

$$(Z-1) = \text{constant}\sqrt{f}, \tag{8.41}$$

where f is the frequency of a K_α X-ray. This equation is **Moseley's law**. For large values of Z, we have approximately

$$Z \approx \text{constant}\sqrt{f}.$$

This prediction can be checked by measuring f for a variety of metal targets. This model is supported if a plot of Z versus $\sqrt{f}$ data (called a **Moseley plot**) is linear. Comparison of model predictions and experimental results, for both the K and L series, is shown in **Figure 8.24**. The data support the model that X-rays are produced when an outer shell electron drops in energy to fill a vacancy in an inner shell.

8.3 Check Your Understanding X-rays are produced by bombarding a metal target with high-energy electrons. If the target is replaced by another with two times the atomic number, what happens to the frequency of X-rays?

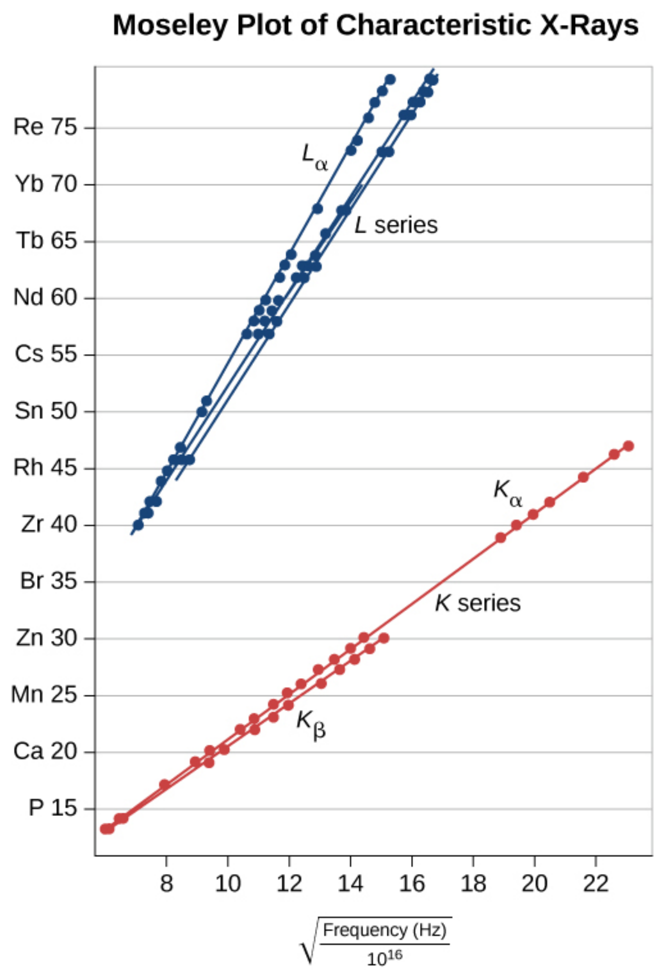


Figure 8.24 A Moseley plot. These data were adapted from Moseley's original data (H. G. J. Moseley, *Philos. Mag*. (6) 77:703, 1914).

Example 8.8

Characteristic X-Ray Energy

Calculate the approximate energy of a K_α X-ray from a tungsten anode in an X-ray tube.

Strategy

Two electrons occupy a filled *K* shell. A vacancy in this shell would leave one electron, so the effective charge for an electron in the *L* shell would be $Z - 1$ rather than *Z*. For tungsten, $Z = 74,$ so the effective charge is 73.

This number can be used to calculate the energy-level difference between the *L* and *K* shells, and, therefore, the energy carried away by a photon in the transition $L \to K$.

Solution

The effective *Z* is 73, so the K_α X-ray energy is given by

$$E_{K_\alpha} = \Delta E = E_\text{i} - E_\text{f} = E_2 - E_1,$$

where

$$E_1 = -\frac{Z^2}{1^2}E_0 = -\frac{73^2}{1}(13.6\text{ eV}) = -72.5\text{ keV}$$

and

$$E_2 = -\frac{Z^2}{2^2}E_0 = -\frac{73^2}{4}(13.6\,\text{eV}) = -18.1\,\text{keV}.$$

Thus,

$$E_{K_\alpha} = -18.1\,\text{keV} - (-72.5\,\text{keV}) = 54.4\,\text{keV}.$$

Significance

This large photon energy is typical of X-rays. X-ray energies become progressively larger for heavier elements because their energy increases approximately as Z^2. An acceleration voltage of more than 50,000 volts is needed to "knock out" an inner electron from a tungsten atom.

X-ray Technology

X-rays have many applications, such as in medical diagnostics (**Figure 8.25**), inspection of luggage at airports (**Figure 8.26**), and even detection of cracks in crucial aircraft components. The most common X-ray images are due to shadows. Because X-ray photons have high energy, they penetrate materials that are opaque to visible light. The more energy an X-ray photon has, the more material it penetrates. The depth of penetration is related to the density of the material, as well as to the energy of the photon. The denser the material, the fewer X-ray photons get through and the darker the shadow. X-rays are effective at identifying bone breaks and tumors; however, overexposure to X-rays can damage cells in biological organisms.

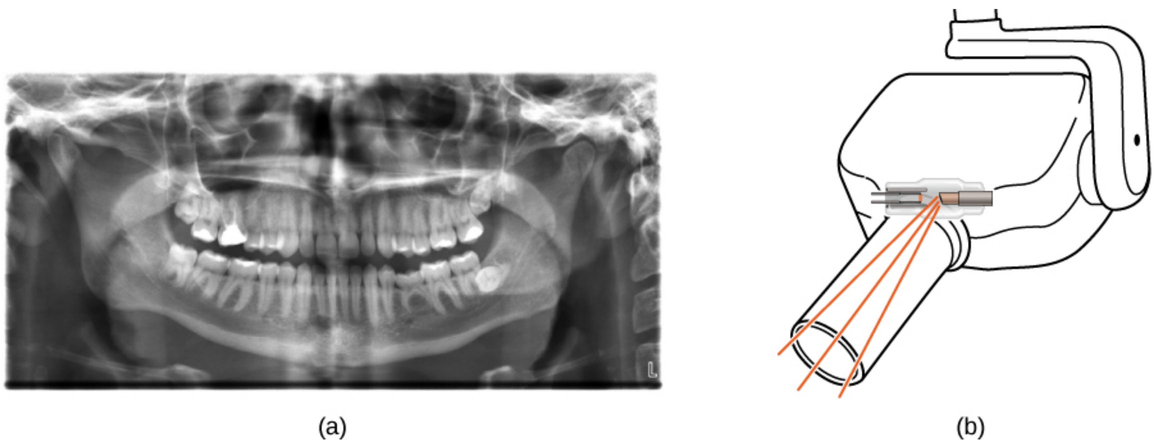


Figure 8.25 (a) An X-ray image of a person's teeth. (b) A typical X-ray machine in a dentist's office produces relatively low-energy radiation to minimize patient exposure. (credit a: modification of work by "Dmitry G"/Wikimedia Commons)

Figure 8.26 An X-ray image of a piece of luggage. The denser the material, the darker the shadow. Object colors relate to material composition—metallic objects show up as blue in this image. (credit: “IDuke”/Wikimedia Commons)

A standard X-ray image provides a two-dimensional view of the object. However, in medical applications, this view does not often provide enough information to draw firm conclusions. For example, in a two-dimensional X-ray image of the body, bones can easily hide soft tissues or organs. The CAT (computed axial tomography) scanner addresses this problem by collecting numerous X-ray images in “slices” throughout the body. Complex computer-image processing of the relative absorption of the X-rays, in different directions, can produce a highly detailed three-dimensional X-ray image of the body.

X-rays can also be used to probe the structures of atoms and molecules. Consider X-rays incident on the surface of a crystalline solid. Some X-ray photons reflect at the surface, and others reflect off the “plane” of atoms just below the surface. Interference between these photons, for different angles of incidence, produces a beautiful image on a screen (**Figure 8.27**). The interaction of X-rays with a solid is called X-ray diffraction. The most famous example using X-ray diffraction is the discovery of the double-helix structure of DNA.

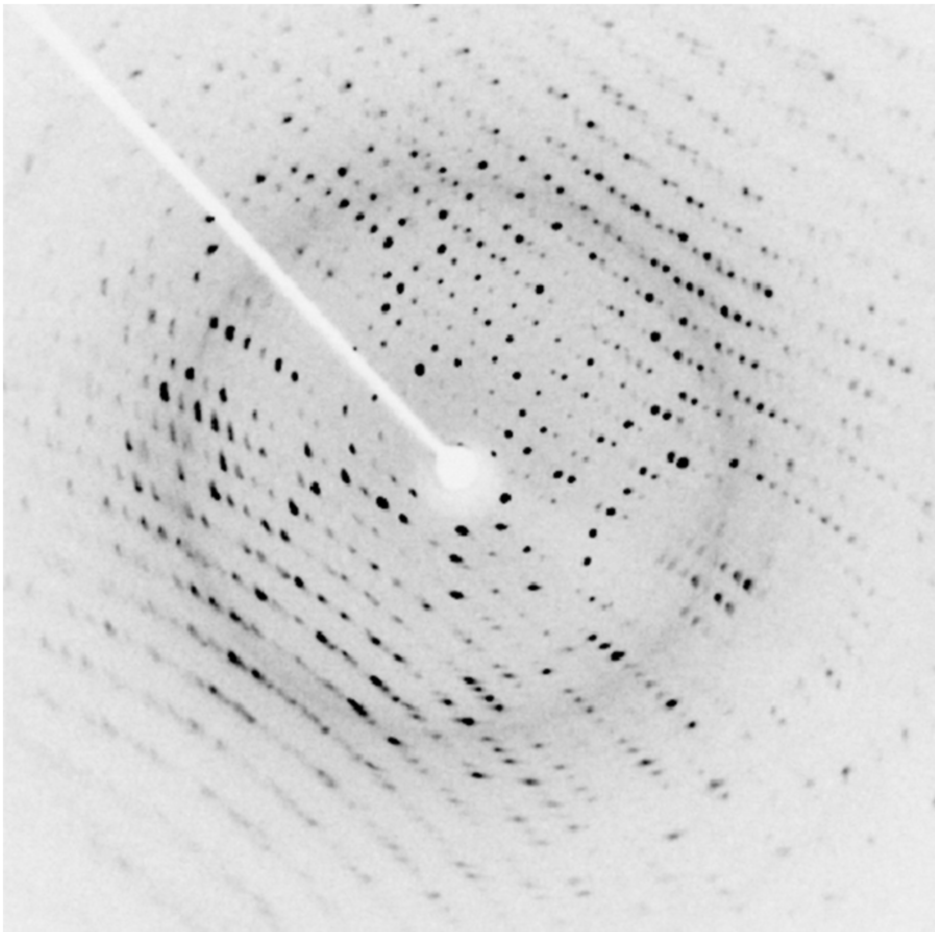


Figure 8.27 X-ray diffraction from the crystal of a protein (hen egg lysozyme) produced this interference pattern. Analysis of the pattern yields information about the structure of the protein. (credit: “Del45”/Wikimedia Commons)

8.6 | Lasers

Learning Objectives

By the end of this section, you will be able to:

- Describe the physical processes necessary to produce laser light
- Explain the difference between coherent and incoherent light
- Describe the application of lasers to a CD and Blu-Ray player

A **laser** is device that emits coherent and monochromatic light. The light is coherent if photons that compose the light are in-phase, and **monochromatic** if the photons have a single frequency (color). When a gas in the laser absorbs radiation, electrons are elevated to different energy levels. Most electrons return immediately to the ground state, but others linger in what is called a **metastable state**. It is possible to place a majority of these atoms in a metastable state, a condition called a **population inversion**.

When a photon of energy disturbs an electron in a metastable state (**Figure 8.28**), the electron drops to the lower-energy level and emits an addition photon, and the two photons proceed off together. This process is called **stimulated emission**. It occurs with relatively high probability when the energy of the incoming photon is equal to the energy difference between the excited and “de-excited” energy levels of the electron ($\Delta E = hf$). Hence, the incoming photon and the photon produced by de-excitation have the same energy, *hf*. These photons encounter more electrons in the metastable state, and the process repeats. The result is a cascade or chain reaction of similar de-excitations. Laser light is coherent because all light waves in laser light share the same frequency (color) and the same phase (any two points of along a line perpendicular to the direction of motion are on the “same part” of the wave”). A schematic diagram of coherent and incoherent light wave pattern is given in **Figure 8.29**.

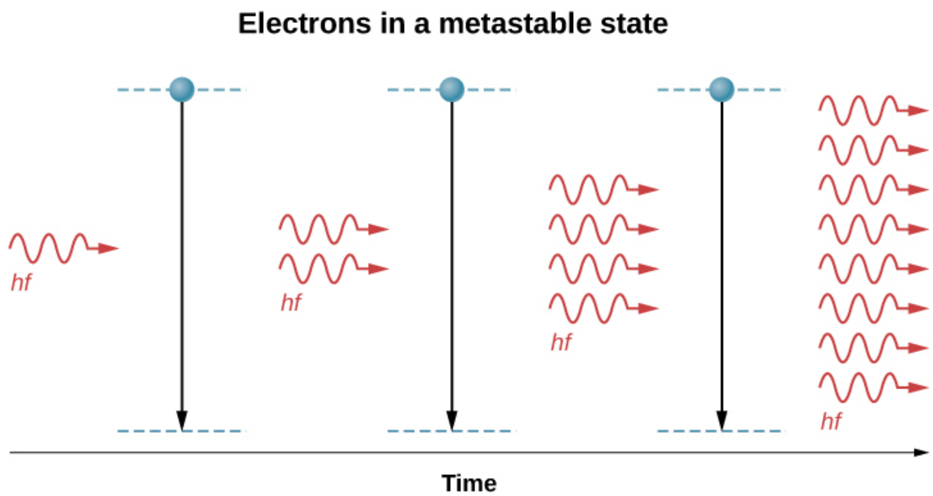


Figure 8.28 The physics of a laser. An incident photon of frequency *f* causes a cascade of photons of the same frequency.

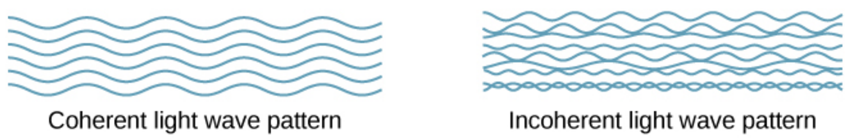


Figure 8.29 A coherent light wave pattern contains light waves of the same frequency and phase. An incoherent light wave pattern contains light waves of different frequencies and phases.

Lasers are used in a wide range of applications, such as in communication (optical fiber phone lines), entertainment (laser light shows), medicine (removing tumors and cauterizing vessels in the retina), and in retail sales (bar code readers). Lasers can also be produced by a large range of materials, including solids (for example, the ruby crystal), gases (helium-gas mixture), and liquids (organic dyes). Recently, a laser was even created using gelatin—an edible laser! Below we discuss two practical applications in detail: CD players and Blu-Ray Players.

CD Player

A CD player reads digital information stored on a compact disc (CD). A CD is 6-inch diameter disc made of plastic that contains small “bumps” and “pits” nears its surface to encode digital or binary data (**Figure 8.30**). The bumps and pits appear along a very thin track that spirals outwards from the center of the disc. The width of the track is smaller than 1/20th the width of a human hair, and the heights of the bumps are even smaller yet.

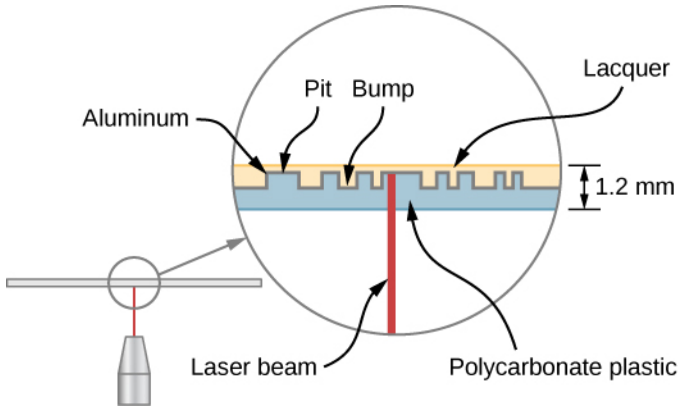


Figure 8.30 A compact disc is a plastic disc that uses bumps near its surface to encode digital information. The surface of the disc contains multiple layers, including a layer of aluminum and one of polycarbonate plastic.

A CD player uses a laser to read this digital information. Laser light is suited to this purpose, because coherent light can be focused onto an incredibly small spot and therefore distinguish between bumps and pits in the CD. After processing by player components (including a diffraction grating, polarizer, and collimator), laser light is focused by a lens onto the CD surface. Light that strikes a bump ("land") is merely reflected, but light that strikes a "pit" destructively interferes, so no light returns (the details of this process are not important to this discussion). Reflected light is interpreted as a "1" and unreflected light is interpreted as a "0." The resulting digital signal is converted into an analog signal, and the analog signal is fed into an amplifier that powers a device such as a pair of headphones. The laser system of a CD player is shown in **Figure 8.31**.

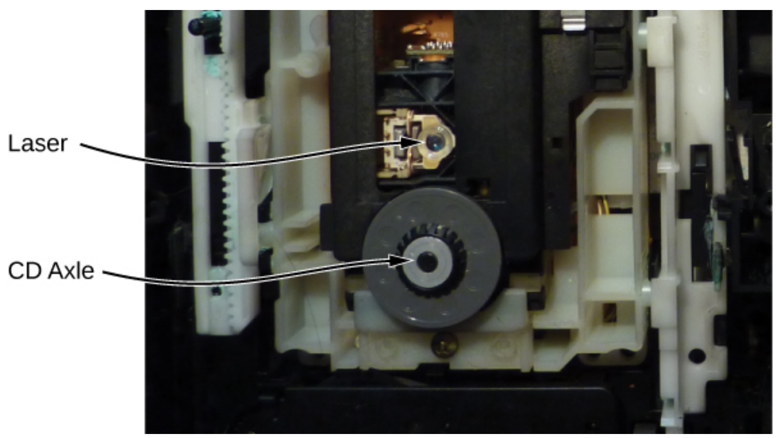


Figure 8.31 A CD player and its laser component.

Blu-Ray Player

Like a CD player, a Blu-Ray player reads digital information (video or audio) stored on a disc, and a laser is used to record this information. The pits on a Blu-Ray disc are much smaller and more closely packed together than for a CD, so much more information can be stored. As a result, the resolving power of the laser must be greater. This is achieved using short wavelength $(\lambda = 405\text{ nm})$ blue laser light—hence, the name "Blu-" Ray. (CDs and DVDs use red laser light.) The different pit sizes and player-hardware configurations of a CD, DVD, and Blu-Ray player are shown in **Figure 8.32**. The pit sizes of a Blu-Ray disk are more than twice as small as the pits on a DVD or CD. Unlike a CD, a Blu-Ray disc store data on a polycarbonate layer, which places the data closer to the lens and avoids readability problems. A hard coating is used to protect the data since it is so close to the surface.

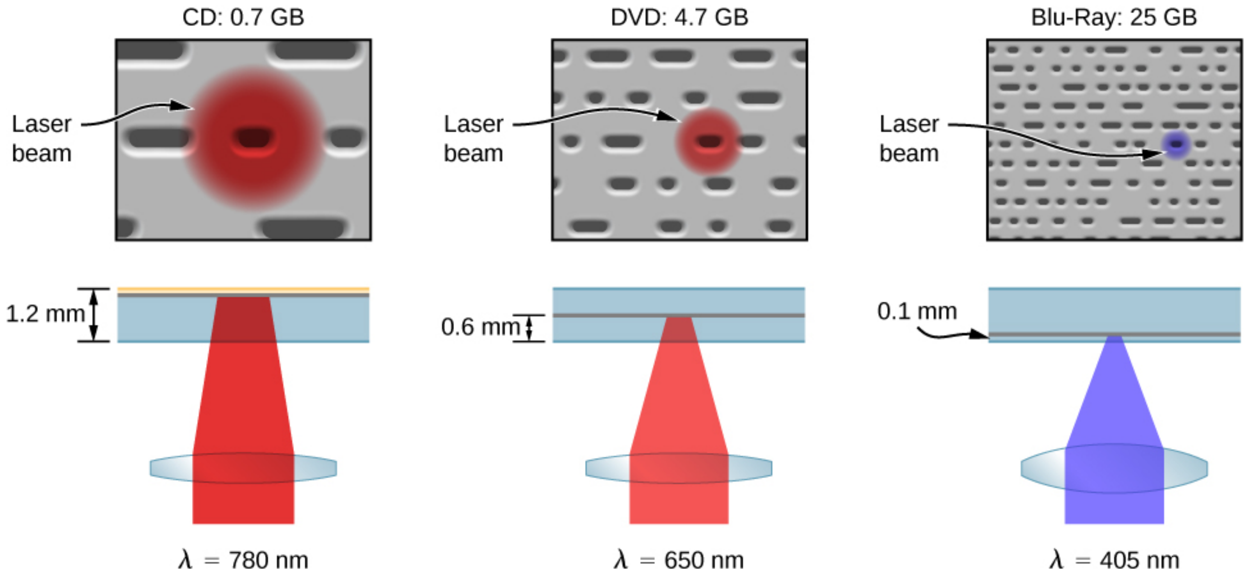


Figure 8.32 Comparison of laser resolution in a CD, DVD, and Blu-Ray Player.

CHAPTER 8 REVIEW

KEY TERMS

angular momentum orbital quantum number (*l*) quantum number associated with the orbital angular momentum of an electron in a hydrogen atom

angular momentum projection quantum number (*m*) quantum number associated with the *z*-component of the orbital angular momentum of an electron in a hydrogen atom

atomic orbital region in space that encloses a certain percentage (usually 90%) of the electron probability

Bohr magneton magnetic moment of an electron, equal to $9.3 \times 10^{-24}\,\text{J/T}$ or $5.8 \times 10^{-5}\,\text{eV/T}$

braking radiation radiation produced by targeting metal with a high-energy electron beam (or radiation produced by the acceleration of any charged particle in a material)

chemical group group of elements in the same column of the periodic table that possess similar chemical properties

coherent light light that consists of photons of the same frequency and phase

covalent bond chemical bond formed by the sharing of electrons between two atoms

electron configuration representation of the state of electrons in an atom, such as $1s^2 2s^1$ for lithium

fine structure detailed structure of atomic spectra produced by spin-orbit coupling

fluorescence radiation produced by the excitation and subsequent, gradual de-excitation of an electron in an atom

hyperfine structure detailed structure of atomic spectra produced by spin-orbit coupling

ionic bond chemical bond formed by the electric attraction between two oppositely charged ions

laser coherent light produced by a cascade of electron de-excitations

magnetic orbital quantum number another term for the angular momentum projection quantum number

magnetogram pictoral representation, or map, of the magnetic activity at the Sun’s surface

metastable state state in which an electron “lingers” in an excited state

monochromatic light that consists of photons with the same frequency

Moseley plot plot of the atomic number versus the square root of X-ray frequency

Moseley’s law relationship between the atomic number and X-ray photon frequency for X-ray production

orbital magnetic dipole moment measure of the strength of the magnetic field produced by the orbital angular momentum of the electron

Pauli’s exclusion principle no two electrons in an atom can have the same values for all four quantum numbers (n, l, m, m_s)

population inversion condition in which a majority of atoms contain electrons in a metastable state

principal quantum number (*n*) quantum number associated with the total energy of an electron in a hydrogen atom

radial probability density function function use to determine the probability of a electron to be found in a spatial interval in *r*

selection rules rules that determine whether atomic transitions are allowed or forbidden (rare)

spin projection quantum number (m_s) quantum number associated with the *z*-component of the spin angular momentum of an electron

spin quantum number (*s*) quantum number associated with the spin angular momentum of an electron

spin-flip transitions atomic transitions between states of an electron-proton system in which the magnetic moments are aligned and not aligned

spin-orbit coupling interaction between the electron magnetic moment and the magnetic field produced by the orbital angular momentum of the electron

stimulated emission when a photon of energy triggers an electron in a metastable state to drop in energy emitting an additional photon

transition metal element that is located in the gap between the first two columns and the last six columns of the table of elements that contains electrons that fill the *d* subshell

valence electron electron in the outer shell of an atom that participates in chemical bonding

Zeeman effect splitting of energy levels by an external magnetic field

KEY EQUATIONS

Orbital angular momentum	$L = \sqrt{l(l+1)}\hbar$
z-component of orbital angular momentum	$L_z = m\hbar$
Radial probability density function	$P(r)dr = \lvert\psi_{n00}\rvert^2 4\pi r^2 dr$
Spin angular momentum	$S = \sqrt{s(s+1)}\hbar$
z-component of spin angular momentum	$S_z = m_s \hbar$
Electron spin magnetic moment	$\vec{\boldsymbol{\mu}}_s = \left(\frac{e}{m_e}\right)\vec{\mathbf{S}}$
Electron orbital magnetic dipole moment	$\vec{\boldsymbol{\mu}} = -\left(\frac{e}{2m_e}\right)\vec{\mathbf{L}}$
Potential energy associated with the magnetic interaction between the orbital magnetic dipole moment and an external magnetic field $\vec{\mathbf{B}}$	$U(\theta) = -\mu_z B = m\mu_{\mathrm{B}} B$
Maximum number of electrons in a subshell of a hydrogen atom	$N = 4l + 2$
Selection rule for atomic transitions in a hydrogen-like atom	$\Delta l = \pm 1$
Moseley's law for X-ray production	$(Z-1) = \text{constant}\sqrt{f}$

SUMMARY

8.1 The Hydrogen Atom

- A hydrogen atom can be described in terms of its wave function, probability density, total energy, and orbital angular momentum.
- The state of an electron in a hydrogen atom is specified by its quantum numbers (*n*, *l*, *m*).
- In contrast to the Bohr model of the atom, the Schrödinger model makes predictions based on probability statements.
- The quantum numbers of a hydrogen atom can be used to calculate important information about the atom.

8.2 Orbital Magnetic Dipole Moment of the Electron

- A hydrogen atom has magnetic properties because the motion of the electron acts as a current loop.
- The energy levels of a hydrogen atom associated with orbital angular momentum are split by an external magnetic field because the orbital angular magnetic moment interacts with the field.

- The quantum numbers of an electron in a hydrogen atom can be used to calculate the magnitude and direction of the orbital magnetic dipole moment of the atom.

8.3 Electron Spin

- The state of an electron in a hydrogen atom can be expressed in terms of five quantum numbers.
- The spin angular momentum quantum of an electron is $= +½$. The spin angular momentum projection quantum number is $m_s = +½$ or $-½$ (spin up or spin down).
- The fine and hyperfine structures of the hydrogen spectrum are explained by magnetic interactions within the atom.

8.4 The Exclusion Principle and the Periodic Table

- Pauli's exclusion principle states that no two electrons in an atom can have all the same quantum numbers.
- The structure of the periodic table of elements can be explained in terms of the total energy, orbital angular momentum, and spin of electrons in an atom.
- The state of an atom can be expressed by its electron configuration, which describes the shells and subshells that are filled in the atom.

8.5 Atomic Spectra and X-rays

- Radiation is absorbed and emitted by atomic energy-level transitions.
- Quantum numbers can be used to estimate the energy, frequency, and wavelength of photons produced by atomic transitions.
- Atomic fluorescence occurs when an electron in an atom is excited several steps above the ground state by the absorption of a high-energy ultraviolet (UV) photon.
- X-ray photons are produced when a vacancy in an inner shell of an atom is filled by an electron from the outer shell of the atom.
- The frequency of X-ray radiation is related to the atomic number Z of an atom.

8.6 Lasers

- Laser light is coherent (monochromatic and "phase linked") light.
- Laser light is produced by population inversion and subsequent de-excitation of electrons in a material (solid, liquid, or gas).
- CD and Blu-Ray players uses lasers to read digital information stored on discs.

CONCEPTUAL QUESTIONS

8.1 The Hydrogen Atom

1. Identify the physical significance of each of the quantum numbers of the hydrogen atom.

2. Describe the ground state of hydrogen in terms of wave function, probability density, and atomic orbitals.

3. Distinguish between Bohr's and Schrödinger's model of the hydrogen atom. In particular, compare the energy and orbital angular momentum of the ground states.

8.2 Orbital Magnetic Dipole Moment of the Electron

4. Explain why spectral lines of the hydrogen atom are split by an external magnetic field. What determines the number and spacing of these lines?

5. A hydrogen atom is placed in a magnetic field. Which of the following quantities are affected? (a) total energy; (b) angular momentum; (c) z-component of angular momentum; (d) polar angle.

6. On what factors does the orbital magnetic dipole moment of an electron depend?

8.3 Electron Spin

7. Explain how a hydrogen atom in the ground state $(l = 0)$ can interact magnetically with an external magnetic field.

8. Compare orbital angular momentum with spin angular momentum of an electron in the hydrogen atom.

9. List all the possible values of s and m_s for an electron. Are there particles for which these values are different?

10. Are the angular momentum vectors $\vec{\mathbf{L}}$ and $\vec{\mathbf{S}}$ necessarily aligned?

11. What is spin-orbit coupling?

8.4 The Exclusion Principle and the Periodic Table

12. What is Pauli's exclusion principle? Explain the importance of this principle for the understanding of atomic structure and molecular bonding.

13. Compare the electron configurations of the elements in the same column of the periodic table.

14. Compare the electron configurations of the elements that belong in the same row of the periodic table of elements.

8.5 Atomic Spectra and X-rays

15. Atomic and molecular spectra are discrete. What does discrete mean, and how are discrete spectra related to the quantization of energy and electron orbits in atoms and molecules?

16. Discuss the process of the absorption of light by matter in terms of the atomic structure of the absorbing medium.

17. NGC1763 is an emission nebula in the Large Magellanic Cloud just outside our Milky Way Galaxy. Ultraviolet light from hot stars ionize the hydrogen atoms in the nebula. As protons and electrons recombine, light in the visible range is emitted. Compare the energies of the photons involved in these two transitions.

18. Why are X-rays emitted only for electron transitions to inner shells? What type of photon is emitted for transitions between outer shells?

19. How do the allowed orbits for electrons in atoms differ from the allowed orbits for planets around the sun?

8.6 Lasers

20. Distinguish between coherent and monochromatic light.

21. Why is a metastable state necessary for the production of laser light?

22. How does light from an incandescent light bulb differ from laser light?

23. How is a Blu-Ray player able to read more information that a CD player?

24. What are the similarities and differences between a CD player and a Blu-Ray player?

PROBLEMS

8.1 The Hydrogen Atom

25. The wave function is evaluated at rectangular coordinates (x, y, z) = (2, 1, 1) in arbitrary units. What are the spherical coordinates of this position?

26. If an atom has an electron in the $n = 5$ state with $m = 3$, what are the possible values of l?

27. What are the possible values of m for an electron in the $n = 4$ state?

28. What, if any, constraints does a value of $m = 1$ place on the other quantum numbers for an electron in an atom?

29. What are the possible values of m for an electron in the $n = 4$ state?

30. (a) How many angles can L make with the z-axis for an $l = 2$ electron? (b) Calculate the value of the smallest angle.

31. The force on an electron is "negative the gradient of the potential energy function." Use this knowledge and **Equation 8.1** to show that the force on the electron in a hydrogen atom is given by Coulomb's force law.

32. What is the total number of states with orbital angular momentum $l = 0$? (Ignore electron spin.)

33. The wave function is evaluated at spherical coordinates $(r, \theta, \phi) = (\sqrt{3}, 45°, 45°)$, where the value of the radial coordinate is given in arbitrary units. What are the rectangular coordinates of this position?

34. Coulomb's force law states that the force between two charged particles is:

$F = k\frac{Qq}{r^2}$. Use this expression to determine the potential energy function.

35. Write an expression for the total number of states with orbital angular momentum *l*.

36. Consider hydrogen in the ground state, ψ_{100}. (a) Use the derivative to determine the radial position for which the probability density, *P*(*r*), is a maximum.

(b) Use the integral concept to determine the average radial position. (This is called the expectation value of the electron's radial position.) Express your answers into terms of the Bohr radius, a_o. Hint: The expectation value is the just average value. (c) Why are these values different?

37. What is the probability that the 1*s* electron of a hydrogen atom is found outside the Bohr radius?

38. How many polar angles are possible for an electron in the $l = 5$ state?

39. What is the maximum number of orbital angular momentum electron states in the $n = 2$ shell of a hydrogen atom? (Ignore electron spin.)

40. What is the maximum number of orbital angular momentum electron states in the $n = 3$ shell of a hydrogen atom? (Ignore electron spin.)

8.2 Orbital Magnetic Dipole Moment of the Electron

41. Find the magnitude of the orbital magnetic dipole moment of the electron in in the 3*p* state. (Express your answer in terms of μ_B.)

42. A current of $I = 2A$ flows through a square-shaped wire with 2-cm side lengths. What is the magnetic moment of the wire?

43. Estimate the ratio of the electron magnetic moment to the *muon* magnetic moment for the same state of orbital angular momentum. (*Hint:* $m_\mu = 105.7 \text{ MeV}/c^2$)

44. Find the magnitude of the orbital magnetic dipole moment of the electron in in the 4*d* state. (Express your answer in terms of μ_B.)

45. For a 3*d* electron in an external magnetic field of 2.50×10^{-3} T, find (a) the current associated with the orbital angular momentum, and (b) the maximum torque.

46. An electron in a hydrogen atom is in the $n = 5$, $l = 4$ state. Find the smallest angle the magnetic moment makes with the *z*-axis. (Express your answer in terms of μ_B.)

47. Find the minimum torque magnitude $|\vec{\tau}|$ that acts on the orbital magnetic dipole of a 3*p* electron in an external magnetic field of 2.50×10^{-3} T.

48. An electron in a hydrogen atom is in 3*p* state. Find the smallest angle the magnetic moment makes with the *z*-axis. (Express your answer in terms of μ_B.)

49. Show that $U = -\vec{\boldsymbol{\mu}} \cdot \vec{\mathbf{B}}$.

(*Hint*: An infinitesimal amount of work is done to align the magnetic moment with the external field. This work rotates the magnetic moment vector through an angle $-d\theta$ (toward the positive *z*-direction), where $d\theta$ is a positive angle change.)

8.3 Electron Spin

50. What is the magnitude of the spin momentum of an electron? (Express you answer in terms of $\hbar$.)

51. What are the possible polar orientations of the spin momentum vector for an electron?

52. For $n = 1$, write all the possible sets of quantum numbers (n, l, m, m_s).

53. A hydrogen atom is placed in an external uniform magnetic field ($B = 200$ T). Calculate the wavelength of light produced in a transition from a spin up to spin down state.

54. If the magnetic field in the preceding problem is quadrupled, what happens to the wavelength of light produced in a transition from a spin up to spin down state?

55. If the magnetic moment in the preceding problem is doubled, what happens to the frequency of light produced in a transition from a spin-up to spin-down state?

56. For $n = 2$, write all the possible sets of quantum numbers (n, l, m, m_s).

8.4 The Exclusion Principle and the Periodic Table

57. (a) How many electrons can be in the $n = 4$ shell?

(b) What are its subshells, and how many electrons can be in each?

58. (a) What is the minimum value of l for a subshell that contains 11 electrons?

(b) If this subshell is in the $n = 5$ shell, what is the spectroscopic notation for this atom?

59. **Unreasonable result.** Which of the following spectroscopic notations are not allowed? (a) $5s^1$ (b) $1d^1$ (c) $4s^3$ (d) $3p^7$ (e) $5g^{15}$. State which rule is violated for each notation that is not allowed.

60. Write the electron configuration for potassium.

61. Write the electron configuration for iron.

62. The valence electron of potassium is excited to a 5*d* state. (a) What is the magnitude of the electron's orbital angular momentum? (b) How many states are possible along a chosen direction?

63. (a) If one subshell of an atom has nine electrons in it, what is the minimum value of *l*? (b) What is the spectroscopic notation for this atom, if this subshell is part of the $n = 3$ shell?

64. Write the electron configuration for magnesium.

65. Write the electron configuration for carbon.

66. The magnitudes of the resultant spins of the electrons of the elements B through Ne when in the ground state are: $\sqrt{3}\hbar/2$, $\sqrt{2}\hbar$, $\sqrt{15}\hbar/2$, $\sqrt{2}\hbar$, $\sqrt{3}\hbar/2$, and 0, respectively. Argue that these spins are consistent with Hund's rule.

8.5 Atomic Spectra and X-rays

67. What is the minimum frequency of a photon required to ionize: (a) a He^+ ion in its ground state? (b) A Li^{2+} ion in its first excited state?

68. The ion Li^{2+} makes an atomic transition from an $n = 4$ state to an $n = 2$ state. (a) What is the energy of the photon emitted during the transition? (b) What is the wavelength of the photon?

69. The red light emitted by a ruby laser has a wavelength of 694.3 nm. What is the difference in energy between the initial state and final state corresponding to the emission of the light?

70. The yellow light from a sodium-vapor street lamp is produced by a transition of sodium atoms from a 3*p* state to a *3s* state. If the difference in energies of those two states is 2.10 eV, what is the wavelength of the yellow light?

71. Estimate the wavelength of the K_α X-ray from calcium.

72. Estimate the frequency of the K_α X-ray from cesium.

73. X-rays are produced by striking a target with a beam of electrons. Prior to striking the target, the electrons are accelerated by an electric field through a potential energy difference:

$$\Delta U = -e\Delta V,$$

where *e* is the charge of an electron and ΔV is the voltage difference. If $\Delta V = 15{,}000$ volts, what is the minimum wavelength of the emitted radiation?

74. For the preceding problem, what happens to the minimum wavelength if the voltage across the X-ray tube is doubled?

75. Suppose the experiment in the preceding problem is conducted with muons. What happens to the minimum wavelength?

76. An X-ray tube accelerates an electron with an applied voltage of 50 kV toward a metal target. (a) What is the shortest-wavelength X-ray radiation generated at the target? (b) Calculate the photon energy in eV. (c) Explain the relationship of the photon energy to the applied voltage.

77. A color television tube generates some X-rays when its electron beam strikes the screen. What is the shortest wavelength of these X-rays, if a 30.0-kV potential is used to accelerate the electrons? (Note that TVs have shielding to prevent these X-rays from exposing viewers.)

78. An X-ray tube has an applied voltage of 100 kV. (a) What is the most energetic X-ray photon it can produce? Express your answer in electron volts and joules. (b) Find the wavelength of such an X-ray.

79. The maximum characteristic X-ray photon energy comes from the capture of a free electron into a *K* shell vacancy. What is this photon energy in keV for tungsten, assuming that the free electron has no initial kinetic energy?

80. What are the approximate energies of the K_α and K_β X-rays for copper?

81. Compare the X-ray photon wavelengths for copper and gold.

82. The approximate energies of the K_α and K_β X-rays for copper are $E_{K_\alpha} = 8.00\,\text{keV}$ and $E_{K_\beta} = 9.48\,\text{keV}$, respectively. Determine the ratio of X-ray frequencies of gold to copper, then use this value to estimate the corresponding energies of K_α and K_β X-rays for gold.

8.6 Lasers

83. A carbon dioxide laser used in surgery emits infrared radiation with a wavelength of $10.6\,\mu\text{m}$. In 1.00 ms, this laser raised the temperature of $1.00\,\text{cm}^3$ of flesh to $100\,°\text{C}$ and evaporated it. (a) How many photons were required? You may assume that flesh has the same heat of vaporization as water. (b) What was the minimum power output during the flash?

84. An excimer laser used for vision correction emits UV radiation with a wavelength of 193 nm. (a) Calculate the photon energy in eV. (b) These photons are used to evaporate corneal tissue, which is very similar to water in its properties. Calculate the amount of energy needed per molecule of water to make the phase change from liquid to gas. That is, divide the heat of vaporization in kJ/kg by the number of water molecules in a kilogram. (c) Convert this to eV and compare to the photon energy. Discuss the implications.

ADDITIONAL PROBLEMS

85. For a hydrogen atom in an excited state with principal quantum number *n*, show that the smallest angle that the orbital angular momentum vector can make with respect to the *z*-axis is $\theta = \cos^{-1}\left(\sqrt{\frac{n-1}{n}}\right)$.

86. What is the probability that the 1*s* electron of a hydrogen atom is found between $r = 0$ and $r = \infty$?

87. Sketch the potential energy function of an electron in a hydrogen atom. (a) What is the value of this function at $r = 0$? in the limit that $r = \infty$? (b) What is unreasonable or inconsistent with the former result?

88. Find the value of l, the orbital angular momentum quantum number, for the Moon around Earth.

89. Show that the maximum number of orbital angular momentum electron states in the *n*th shell of an atom is n^2. (Ignore electron spin.) (*Hint:* Make a table of the total number of orbital angular momentum states for each shell and find the pattern.)

90. What is the magnitude of an electron magnetic moment?

91. What is the maximum number of electron states in the $n = 5$ shell?

92. A ground-state hydrogen atom is placed in a uniform magnetic field, and a photon is emitted in the transition from a spin-up to spin-down state. The wavelength of the photon is $168\,\mu\text{m}$. What is the strength of the magnetic field?

93. Show that the maximum number of electron states in the *n*th shell of an atom is $2n^2$.

94. The valence electron of chlorine is excited to a 3*p* state. (a) What is the magnitude of the electron's orbital angular momentum? (b) What are possible values for the *z*-component of angular measurement?

95. Which of the following notations are allowed (that is, which violate none of the rules regarding values of quantum numbers)? (a) $1s^1$; (b) $1d^3$; (c) $4s^2$; (d) $3p^7$; (e) $6h^{20}$

96. The ion Be^{3+} makes an atomic transition from an $n = 3$ state to an $n = 2$ state. (a) What is the energy of the photon emitted during the transition? (b) What is the wavelength of the photon?

97. The maximum characteristic X-ray photon energy comes from the capture of a free electron into a *K* shell vacancy. What is this photon frequency for tungsten, assuming that the free electron has no initial kinetic energy?

98. Derive an expression for the ratio of X-ray photon frequency for two elements with atomic numbers Z_1 and Z_2.

99. Compare the X-ray photon wavelengths for copper and silver.

100. (a) What voltage must be applied to an X-ray tube to obtain 0.0100-fm-wavelength X-rays for use in exploring the details of nuclei? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

101. A student in a physics laboratory observes a hydrogen spectrum with a diffraction grating for the purpose of measuring the wavelengths of the emitted radiation. In the spectrum, she observes a yellow line and finds its wavelength to be 589 nm. (a) Assuming that this is part of the Balmer series, determine n_i, the principal quantum number of the initial state. (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

APPENDIX A | UNITS

Quantity	Common Symbol	Unit	Unit in Terms of Base SI Units
Acceleration	$\vec{\mathbf{a}}$	m/s^2	m/s^2
Amount of substance	*n*	**mole**	mol
Angle	θ, ϕ	radian (rad)	
Angular acceleration	$\vec{\alpha}$	rad/s^2	s^{-2}
Angular frequency	ω	rad/s	s^{-1}
Angular momentum	$\vec{\mathbf{L}}$	$kg \cdot m^2/s$	$kg \cdot m^2/s$
Angular velocity	$\vec{\omega}$	rad/s	s^{-1}
Area	*A*	m^2	m^2
Atomic number	*Z*		
Capacitance	*C*	farad (F)	$A^2 \cdot s^4/kg \cdot m^2$
Charge	*q*, *Q*, *e*	coulomb (C)	$A \cdot s$
Charge density:			
Line	λ	C/m	$A \cdot s/m$
Surface	σ	C/m^2	$A \cdot s/m^2$
Volume	ρ	C/m^3	$A \cdot s/m^3$
Conductivity	σ	$1/\Omega \cdot m$	$A^2 \cdot s^3/kg \cdot m^3$
Current	*I*	**ampere**	A
Current density	$\vec{\mathbf{J}}$	A/m^2	A/m^2
Density	ρ	kg/m^3	kg/m^3
Dielectric constant	κ		
Electric dipole moment	$\vec{\mathbf{p}}$	$C \cdot m$	$A \cdot s \cdot m$
Electric field	$\vec{\mathbf{E}}$	N/C	$kg \cdot m/A \cdot s^3$
Electric flux	Φ	$N \cdot m^2/C$	$kg \cdot m^3/A \cdot s^3$
Electromotive force	ε	volt (V)	$kg \cdot m^2/A \cdot s^3$
Energy	*E*,*U*,*K*	joule (J)	$kg \cdot m^2/s^2$
Entropy	*S*	J/K	$kg \cdot m^2/s^2 \cdot K$

Table A1 Units Used in Physics (Fundamental units in bold)

Quantity	Common Symbol	Unit	Unit in Terms of Base SI Units
Force	$\vec{\mathbf{F}}$	newton (N)	$\text{kg} \cdot \text{m/s}^2$
Frequency	*f*	hertz (Hz)	s^{-1}
Heat	*Q*	joule (J)	$\text{kg} \cdot \text{m}^2/\text{s}^2$
Inductance	*L*	henry (H)	$\text{kg} \cdot \text{m}^2/\text{A}^2 \cdot \text{s}^2$
Length:	ℓ, L	**meter**	m
Displacement	Δx, $\Delta \vec{\mathbf{r}}$		
Distance	*d*, *h*		
Position	x, y, z, $\vec{\mathbf{r}}$		
Magnetic dipole moment	$\vec{\mu}$	$\text{N} \cdot \text{J/T}$	$\text{A} \cdot \text{m}^2$
Magnetic field	$\vec{\mathbf{B}}$	$\text{tesla(T)} = \left(\text{Wb/m}^2\right)$	$\text{kg/A} \cdot \text{s}^2$
Magnetic flux	Φ_m	weber (Wb)	$\text{kg} \cdot \text{m}^2/\text{A} \cdot \text{s}^2$
Mass	*m*, *M*	**kilogram**	kg
Molar specific heat	*C*	$\text{J/mol} \cdot \text{K}$	$\text{kg} \cdot \text{m}^2/\text{s}^2 \cdot \text{mol} \cdot \text{K}$
Moment of inertia	*I*	$\text{kg} \cdot \text{m}^2$	$\text{kg} \cdot \text{m}^2$
Momentum	$\vec{\mathbf{p}}$	$\text{kg} \cdot \text{m/s}$	$\text{kg} \cdot \text{m/s}$
Period	*T*	s	s
Permeability of free space	μ_0	$\text{N/A}^2 = \text{(H/m)}$	$\text{kg} \cdot \text{m/A}^2 \cdot \text{s}^2$
Permittivity of free space	ε_0	$\text{C}^2/\text{N} \cdot \text{m}^2 = \text{(F/m)}$	$\text{A}^2 \cdot \text{s}^4/\text{kg} \cdot \text{m}^3$
Potential	*V*	$\text{volt(V)} = \text{(J/C)}$	$\text{kg} \cdot \text{m}^2/\text{A} \cdot \text{s}^3$
Power	*P*	$\text{watt(W)} = \text{(J/s)}$	$\text{kg} \cdot \text{m}^2/\text{s}^3$
Pressure	*p*	$\text{pascal(Pa)} = \left(\text{N/m}^2\right)$	$\text{kg/m} \cdot \text{s}^2$
Resistance	*R*	$\text{ohm}(\Omega) = \text{(V/A)}$	$\text{kg} \cdot \text{m}^2/\text{A}^2 \cdot \text{s}^3$
Specific heat	*c*	$\text{J/kg} \cdot \text{K}$	$\text{m}^2/\text{s}^2 \cdot \text{K}$
Speed	v	m/s	m/s
Temperature	*T*	**kelvin**	K
Time	*t*	**second**	s
Torque	$\vec{\tau}$	$\text{N} \cdot \text{m}$	$\text{kg} \cdot \text{m}^2/\text{s}^2$

Table A1 Units Used in Physics (Fundamental units in bold)

Quantity	Common Symbol	Unit	Unit in Terms of Base SI Units
Velocity	$\vec{\mathbf{v}}$	m/s	m/s
Volume	*V*	m^3	m^3
Wavelength	λ	m	m
Work	*W*	$\text{joule(J)} = (\text{N} \cdot \text{m})$	$\text{kg} \cdot \text{m}^2/\text{s}^2$

Table A1 Units Used in Physics (Fundamental units in bold)

APPENDIX B | CONVERSION FACTORS

	m	cm	km
1 meter	1	10^2	10^{-3}
1 centimeter	10^{-2}	1	10^{-5}
1 kilometer	10^3	10^5	1
1 inch	2.540×10^{-2}	2.540	2.540×10^{-5}
1 foot	0.3048	30.48	3.048×10^{-4}
1 mile	1609	1.609×10^4	1.609
1 angstrom	10^{-10}		
1 fermi	10^{-15}		
1 light-year			9.460×10^{12}
	in.	**ft**	**mi**
1 meter	39.37	3.281	6.214×10^{-4}
1 centimeter	0.3937	3.281×10^{-2}	6.214×10^{-6}
1 kilometer	3.937×10^4	3.281×10^3	0.6214
1 inch	1	8.333×10^{-2}	1.578×10^{-5}
1 foot	12	1	1.894×10^{-4}
1 mile	6.336×10^4	5280	1

Table B1 Length

Area

$$1\ \text{cm}^2 = 0.155\ \text{in.}^2$$

$$1\ \text{m}^2 = 10^4\ \text{cm}^2 = 10.76\ \text{ft}^2$$

$$1\ \text{in.}^2 = 6.452\ \text{cm}^2$$

$$1\ \text{ft}^2 = 144\ \text{in.}^2 = 0.0929\ \text{m}^2$$

Volume

$$1\ \text{liter} = 1000\ \text{cm}^3 = 10^{-3}\ \text{m}^3 = 0.03531\ \text{ft}^3 = 61.02\ \text{in.}^3$$

$$1\ \text{ft}^3 = 0.02832\ \text{m}^3 = 28.32\ \text{liters} = 7.477\ \text{gallons}$$

$$1\ \text{gallon} = 3.788\ \text{liters}$$

	s	min	h	day	yr
1 second	1	1.667×10^{-2}	2.778×10^{-4}	1.157×10^{-5}	3.169×10^{-8}
1 minute	60	1	1.667×10^{-2}	6.944×10^{-4}	1.901×10^{-6}
1 hour	3600	60	1	4.167×10^{-2}	1.141×10^{-4}
1 day	8.640×10^{4}	1440	24	1	2.738×10^{-3}
1 year	3.156×10^{7}	5.259×10^{5}	8.766×10^{3}	365.25	1

Table B2 Time

	m/s	cm/s	ft/s	mi/h
1 meter/second	1	10^{2}	3.281	2.237
1 centimeter/second	10^{-2}	1	3.281×10^{-2}	2.237×10^{-2}
1 foot/second	0.3048	30.48	1	0.6818
1 mile/hour	0.4470	44.70	1.467	1

Table B3 Speed

Acceleration

$1\ \text{m/s}^2 = 100\ \text{cm/s}^2 = 3.281\ \text{ft/s}^2$

$1\ \text{cm/s}^2 = 0.01\ \text{m/s}^2 = 0.03281\ \text{ft/s}^2$

$1\ \text{ft/s}^2 = 0.3048\ \text{m/s}^2 = 30.48\ \text{cm/s}^2$

$1\ \text{mi/h} \cdot \text{s} = 1.467\ \text{ft/s}^2$

	kg	g	slug	u
1 kilogram	1	10^{3}	6.852×10^{-2}	6.024×10^{26}
1 gram	10^{-3}	1	6.852×10^{-5}	6.024×10^{23}
1 slug	14.59	1.459×10^{4}	1	8.789×10^{27}
1 atomic mass unit	1.661×10^{-27}	1.661×10^{-24}	1.138×10^{-28}	1
1 metric ton	1000			

Table B4 Mass

	N	dyne	lb
1 newton	1	10^{5}	0.2248
1 dyne	10^{-5}	1	2.248×10^{-6}
1 pound	4.448	4.448×10^{5}	1

Table B5 Force

	Pa	$dyne/cm^2$	atm	cmHg	$lb/in.^2$
1 pascal	1	10	9.869×10^{-6}	7.501×10^{-4}	1.450×10^{-4}
1 dyne/ centimeter^2	10^{-1}	1	9.869×10^{-7}	7.501×10^{-5}	1.450×10^{-5}
1 atmosphere	1.013×10^{5}	1.013×10^{6}	1	76	14.70
1 centimeter mercury*	1.333×10^{3}	1.333×10^{4}	1.316×10^{-2}	1	0.1934
1 pound/inch^2	6.895×10^{3}	6.895×10^{4}	6.805×10^{-2}	5.171	1
1 bar	10^5				
1 torr				1 (mmHg)	

***Where the acceleration due to gravity is $9.80665\ m/s^2$ and the temperature is $0°C$**

Table B6 Pressure

	J	erg	ft.lb
1 joule	1	10^7	0.7376
1 erg	10^{-7}	1	7.376×10^{-8}
1 foot-pound	1.356	1.356×10^{7}	1
1 electron-volt	1.602×10^{-19}	1.602×10^{-12}	1.182×10^{-19}
1 calorie	4.186	4.186×10^{7}	3.088
1 British thermal unit	1.055×10^{3}	1.055×10^{10}	7.779×10^{2}
1 kilowatt-hour	3.600×10^{6}		
	eV	**cal**	**Btu**
1 joule	6.242×10^{18}	0.2389	9.481×10^{-4}
1 erg	6.242×10^{11}	2.389×10^{-8}	9.481×10^{-11}
1 foot-pound	8.464×10^{18}	0.3239	1.285×10^{-3}
1 electron-volt	1	3.827×10^{-20}	1.519×10^{-22}
1 calorie	2.613×10^{19}	1	3.968×10^{-3}
1 British thermal unit	6.585×10^{21}	2.520×10^{2}	1

Table B7 Work, Energy, Heat

Power

$1\ W = 1\ J/s$

$1\ hp = 746\ W = 550\ ft \cdot lb/s$

$1\ Btu/h = 0.293\ W$

Angle

$$1 \text{ rad} = 57.30° = 180°/\pi$$

$$1° = 0.01745 \text{ rad} = \pi/180 \text{ rad}$$

$$1 \text{ revolution} = 360° = 2\pi \text{ rad}$$

$$1 \text{ rev/min(rpm)} = 0.1047 \text{ rad/s}$$

APPENDIX C | FUNDAMENTAL CONSTANTS

Quantity	Symbol	Value
Atomic mass unit	u	$1.660\ 538\ 782\ (83) \times 10^{-27}$ kg $931.494\ 028\ (23)$ MeV/c^2
Avogadro's number	N_A	$6.022\ 141\ 79\ (30) \times 10^{23}$ particles/mol
Bohr magneton	$\mu_B = \frac{e\hbar}{2m_e}$	$9.274\ 009\ 15\ (23) \times 10^{-24}$ J/T
Bohr radius	$a_0 = \frac{\hbar^2}{m_e e^2 k_e}$	$5.291\ 772\ 085\ 9\ (36) \times 10^{-11}$ m
Boltzmann's constant	$k_B = \frac{R}{N_A}$	$1.380\ 650\ 4\ (24) \times 10^{-23}$ J/K
Compton wavelength	$\lambda_C = \frac{h}{m_e c}$	$2.426\ 310\ 217\ 5\ (33) \times 10^{-12}$ m
Coulomb constant	$k_e = \frac{1}{4\pi\varepsilon_0}$	$8.987\ 551\ 788... \times 10^9\ \text{N} \cdot \text{m}^2/\text{C}^2$ (exact)
Deuteron mass	m_d	$3.343\ 583\ 20\ (17) \times 10^{-27}$ kg $2.013\ 553\ 212\ 724(78)$ u $1875.612\ 859$ MeV/c^2
Electron mass	m_e	$9.109\ 382\ 15\ (45) \times 10^{-31}$ kg $5.485\ 799\ 094\ 3(23) \times 10^{-4}$ u $0.510\ 998\ 910\ (13)$ MeV/c^2
Electron volt	eV	$1.602\ 176\ 487\ (40) \times 10^{-19}$ J
Elementary charge	e	$1.602\ 176\ 487\ (40) \times 10^{-19}$ C
Gas constant	R	$8.314\ 472\ (15)$ J/mol · K
Gravitational constant	G	$6.674\ 28\ (67) \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2$

Table C1 Fundamental Constants *Note:* These constants are the values recommended in 2006 by CODATA, based on a least-squares adjustment of data from different measurements. The numbers in parentheses for the values represent the uncertainties of the last two digits.

Quantity	Symbol	Value
Neutron mass	m_n	$1.674\ 927\ 211\ (84) \times 10^{-27}$ kg $1.008\ 664\ 915\ 97\ (43)$ u $939.565\ 346\ (23)\ \text{MeV}/c^2$
Nuclear magneton	$\mu_n = \frac{e\hbar}{2m_p}$	$5.050\ 783\ 24\ (13) \times 10^{-27}$ J/T
Permeability of free space	μ_0	$4\pi \times 10^{-7}\ \text{T} \cdot \text{m/A}$(exact)
Permittivity of free space	$\varepsilon_0 = \frac{1}{\mu_0 c^2}$	$8.854\ 187\ 817... \times 10^{-12}\ \text{C}^2/\text{N} \cdot \text{m}^2$ (exact)
Planck's constant	h $\hbar = \frac{h}{2\pi}$	$6.626\ 068\ 96\ (33) \times 10^{-34}$ J · s $1.054\ 571\ 628\ (53) \times 10^{-34}$ J · s
Proton mass	m_p	$1.672\ 621\ 637\ (83) \times 10^{-27}$ kg $1.007\ 276\ 466\ 77\ (10)$ u $938.272\ 013\ (23)\ \text{MeV}/c^2$
Rydberg constant	R_H	$1.097\ 373\ 156\ 852\ 7\ (73) \times 10^7\ \text{m}^{-1}$
Speed of light in vacuum	c	$2.997\ 924\ 58 \times 10^8$ m/s (exact)

Table C1 Fundamental Constants *Note:* These constants are the values recommended in 2006 by CODATA, based on a least-squares adjustment of data from different measurements. The numbers in parentheses for the values represent the uncertainties of the last two digits.

Useful combinations of constants for calculations:

$$hc = 12{,}400\ \text{eV} \cdot \text{Å} = 1240\ \text{eV} \cdot \text{nm} = 1240\ \text{MeV} \cdot \text{fm}$$

$$\hbar c = 1973\ \text{eV} \cdot \text{Å} = 197.3\ \text{eV} \cdot \text{nm} = 197.3\ \text{MeV} \cdot \text{fm}$$

$$k_e e^2 = 14.40\ \text{eV} \cdot \text{Å} = 1.440\ \text{eV} \cdot \text{nm} = 1.440\ \text{MeV} \cdot \text{fm}$$

$$k_\text{B} T = 0.02585\ \text{eV at } T = 300\ \text{K}$$

APPENDIX D | ASTRONOMICAL DATA

Celestial Object	Mean Distance from Sun (million km)	Period of Revolution (d = days) (y = years)	Period of Rotation at Equator	Eccentricity of Orbit
Sun	–	–	27 d	–
Mercury	57.9	88 d	59 d	0.206
Venus	108.2	224.7 d	243 d	0.007
Earth	149.6	365.26 d	23 h 56 min 4 s	0.017
Mars	227.9	687 d	24 h 37 min 23 s	0.093
Jupiter	778.4	11.9 y	9 h 50 min 30 s	0.048
Saturn	1426.7	29.5 6	10 h 14 min	0.054
Uranus	2871.0	84.0 y	17 h 14 min	0.047
Neptune	4498.3	164.8 y	16 h	0.009
Earth's Moon	149.6 (0.386 from Earth)	27.3 d	27.3 d	0.055
Celestial Object	**Equatorial Diameter (km)**	**Mass (Earth = 1)**	**Density (g/cm^3)**	
Sun	1,392,000	333,000.00	1.4	
Mercury	4879	0.06	5.4	
Venus	12,104	0.82	5.2	
Earth	12,756	1.00	5.5	
Mars	6794	0.11	3.9	
Jupiter	142,984	317.83	1.3	
Saturn	120,536	95.16	0.7	
Uranus	51,118	14.54	1.3	
Neptune	49,528	17.15	1.6	
Earth's Moon	3476	0.01	3.3	

Table D1 Astronomical Data

Other Data:

Mass of Earth: 5.97×10^{24} kg

Mass of the Moon: 7.36×10^{22} kg

Mass of the Sun: 1.99×10^{30} kg

APPENDIX E | MATHEMATICAL FORMULAS

Quadratic formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Triangle of base b and height h	**Area $= \frac{1}{2}bh$**	
Circle of radius r	Circumference $= 2\pi r$	Area $= \pi r^2$
Sphere of radius r	Surface area $= 4\pi r^2$	Volume $= \frac{4}{3}\pi r^3$
Cylinder of radius r and height h	Area of curved surface $= 2\pi rh$	Volume $= \pi r^2 h$

Table E1 Geometry

Trigonometry

Trigonometric Identities

1. $\sin\theta = 1/\csc\theta$
2. $\cos\theta = 1/\sec\theta$
3. $\tan\theta = 1/\cot\theta$
4. $\sin\left(90^0 - \theta\right) = \cos\theta$
5. $\cos\left(90^0 - \theta\right) = \sin\theta$
6. $\tan\left(90^0 - \theta\right) = \cot\theta$
7. $\sin^2\theta + \cos^2\theta = 1$
8. $\sec^2\theta - \tan^2\theta = 1$
9. $\tan\theta = \sin\theta/\cos\theta$
10. $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
11. $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
12. $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$
13. $\sin 2\theta = 2\sin\theta\cos\theta$
14. $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

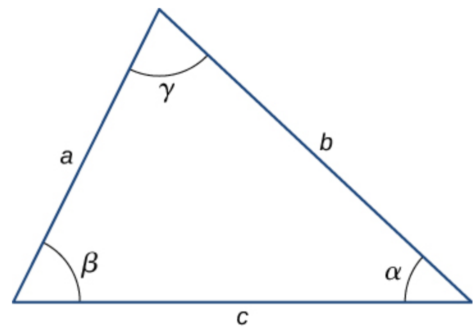
15. $\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$

16. $\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$

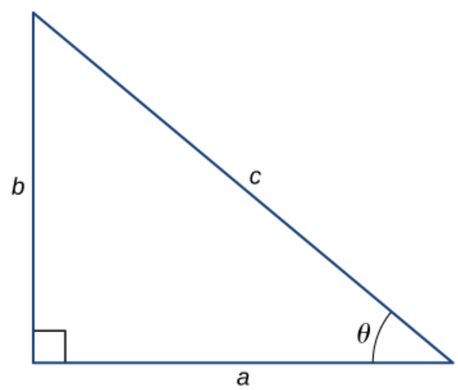
Triangles

1. Law of sines: $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$

2. Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\gamma$



3. Pythagorean theorem: $a^2 + b^2 = c^2$



Series expansions

1. Binomial theorem: $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \cdots$

2. $(1 \pm x)^n = 1 \pm \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} \pm \cdots \left(x^2 < 1\right)$

3. $(1 \pm x)^{-n} = 1 \mp \frac{nx}{1!} + \frac{n(n+1)x^2}{2!} \mp \cdots \left(x^2 < 1\right)$

4. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

5. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

6. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$

7. $e^x = 1 + x + \frac{x^2}{2!} + \cdots$

8. $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots(|x| < 1)$

Derivatives

1. $\frac{d}{dx}[af(x)] = a\frac{d}{dx}f(x)$
2. $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
3. $\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$
4. $\frac{d}{dx}f(u) = \left[\frac{d}{du}f(u)\right]\frac{du}{dx}$
5. $\frac{d}{dx}x^m = mx^{m-1}$
6. $\frac{d}{dx}\sin x = \cos x$
7. $\frac{d}{dx}\cos x = -\sin x$
8. $\frac{d}{dx}\tan x = \sec^2 x$
9. $\frac{d}{dx}\cot x = -\csc^2 x$
10. $\frac{d}{dx}\sec x = \tan x \sec x$
11. $\frac{d}{dx}\csc x = -\cot x \csc x$
12. $\frac{d}{dx}e^x = e^x$
13. $\frac{d}{dx}\ln x = \frac{1}{x}$
14. $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
15. $\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$
16. $\frac{d}{dx}\tan^{-1} x = -\frac{1}{1+x^2}$

Integrals

1. $\int af(x)dx = a\int f(x)dx$
2. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$
3. $\int x^m dx = \frac{x^{m+1}}{m+1} \, (m \neq -1)$
   $= \ln x (m = -1)$
4. $\int \sin x\, dx = -\cos x$
5. $\int \cos x\, dx = \sin x$
6. $\int \tan x\, dx = \ln|\sec x|$

7. $\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$

8. $\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$

9. $\int \sin ax \cos ax\,dx = -\frac{\cos 2ax}{4a}$

10. $\int e^{ax}\,dx = \frac{1}{a}e^{ax}$

11. $\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax - 1)$

12. $\int \ln ax\,dx = x \ln ax - x$

13. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

14. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{x + a}{x - a}\right|$

15. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\frac{x}{a}$

16. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}$

17. $\int \sqrt{a^2 + x^2}\,dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$

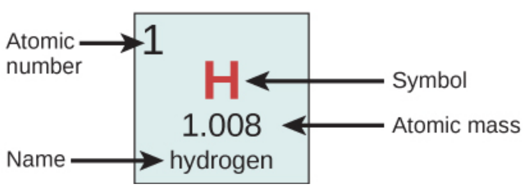
18. $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

APPENDIX F | CHEMISTRY

Periodic Table of the Elements

Period	Group 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1 H 1.008 hydrogen																	2 He 4.003 helium
2	3 Li 6.94 lithium	4 Be 9.012 beryllium											5 B 10.81 boron	6 C 12.01 carbon	7 N 14.01 nitrogen	8 O 16.00 oxygen	9 F 19.00 fluorine	10 Ne 20.18 neon
3	11 Na 22.99 sodium	12 Mg 24.31 magnesium											13 Al 26.98 aluminum	14 Si 28.09 silicon	15 P 30.97 phosphorus	16 S 32.06 sulfur	17 Cl 35.45 chlorine	18 Ar 39.95 argon
4	19 K 39.10 potassium	20 Ca 40.08 calcium	21 Sc 44.96 scandium	22 Ti 47.87 titanium	23 V 50.94 vanadium	24 Cr 52.00 chromium	25 Mn 54.94 manganese	26 Fe 55.85 iron	27 Co 58.93 cobalt	28 Ni 58.69 nickel	29 Cu 63.55 copper	30 Zn 65.38 zinc	31 Ga 69.72 gallium	32 Ge 72.63 germanium	33 As 74.92 arsenic	34 Se 78.97 selenium	35 Br 79.90 bromine	36 Kr 83.80 krypton
5	37 Rb 85.47 rubidium	38 Sr 87.62 strontium	39 Y 88.91 yttrium	40 Zr 91.22 zirconium	41 Nb 92.91 niobium	42 Mo 95.95 molybdenum	43 Tc [97] technetium	44 Ru 101.1 ruthenium	45 Rh 102.9 rhodium	46 Pd 106.4 palladium	47 Ag 107.9 silver	48 Cd 112.4 cadmium	49 In 114.8 indium	50 Sn 118.7 tin	51 Sb 121.8 antimony	52 Te 127.6 tellurium	53 I 126.9 iodine	54 Xe 131.3 xenon
6	55 Cs 132.9 cesium	56 Ba 137.3 barium	57-71 La–Lu *	72 Hf 178.5 hafnium	73 Ta 180.9 tantalum	74 W 183.8 tungsten	75 Re 186.2 rhenium	76 Os 190.2 osmium	77 Ir 192.2 iridium	78 Pt 195.1 platinum	79 Au 197.0 gold	80 Hg 200.6 mercury	81 Tl 204.4 thallium	82 Pb 207.2 lead	83 Bi 209.0 bismuth	84 Po [209] polonium	85 At [210] astatine	86 Rn [222] radon
7	87 Fr [223] francium	88 Ra [226] radium	89-103 Ac–Lr **	104 Rf [267] rutherfordium	105 Db [270] dubnium	106 Sg [271] seaborgium	107 Bh [270] bohrium	108 Hs [277] hassium	109 Mt [276] meitnerium	110 Ds [281] darmstadtium	111 Rg [282] roentgenium	112 Cn [285] copernicium	113 Uut [285] ununtrium	114 Fl [289] flerovium	115 Uup [288] ununpentium	116 Lv [293] livermorium	117 Uus [294] ununseptium	118 Uuo [294] ununoctium

*	57 La 138.9 lanthanum	58 Ce 140.1 cerium	59 Pr 140.9 praseodymium	60 Nd 144.2 neodymium	61 Pm [145] promethium	62 Sm 150.4 samarium	63 Eu 152.0 europium	64 Gd 157.3 gadolinium	65 Tb 158.9 terbium	66 Dy 162.5 dysprosium	67 Ho 164.9 holmium	68 Er 167.3 erbium	69 Tm 168.9 thulium	70 Yb 173.1 ytterbium	71 Lu 175.0 lutetium
**	89 Ac [227] actinium	90 Th 232.0 thorium	91 Pa 231.0 protactinium	92 U 238.0 uranium	93 Np [237] neptunium	94 Pu [244] plutonium	95 Am [243] americium	96 Cm [247] curium	97 Bk [247] berkelium	98 Cf [251] californium	99 Es [252] einsteinium	100 Fm [257] fermium	101 Md [258] mendelevium	102 No [259] nobelium	103 Lr [262] lawrencium



Color Code

Metal	Solid
Metalloid	Liquid
Nonmetal	Gas

APPENDIX G | THE GREEK ALPHABET

Name	Capital	Lowercase	Name	Capital	Lowercase
Alpha	A	α	Nu	N	ν
Beta	B	β	Xi	Ξ	ξ
Gamma	Γ	γ	Omicron	O	o
Delta	Δ	δ	Pi	Π	π
Epsilon	E	ε	Rho	P	ρ
Zeta	Z	ζ	Sigma	Σ	σ
Eta	H	η	Tau	T	τ
Theta	Θ	θ	Upsilon	Υ	υ
Iota	I	ι	Phi	Φ	ϕ
Kappa	K	κ	Chi	X	χ
Lambda	Λ	λ	Psi	Ψ	ψ
Mu	M	μ	Omega	Ω	ω

Table G1 The Greek Alphabet

67. 1.0 m
69. 1.2 cm or closer
71. no
73. 0.120 nm
75. $4.51°$
77. $13.2°$

ADDITIONAL PROBLEMS

79. a. 2.2 mm; b. $0.172°$, second-order yellow and third-order violet coincide
81. 2.2 km
83. 1.3 cm
85. a. 0.28 mm; b. 0.28 m; c. 280 m; d. 113 km
87. 33 m
89. a. vertically; b. $\pm 20°$, $\pm 44°$; c. 0, $\pm 31°$, $\pm 60°$; d. 89 cm; e. 71 cm
91. 0.98 cm
93. $I/I_0 = 0.041$
95. 340 nm
97. a. 0.082 rad and 0.087 rad; b. 480 nm and 660 nm
99. two orders
101. yes and N/A
103. 600 nm
105. a. $3.4 \times 10^{-5}\,°$; b. $51°$
107. 0.63 m
109. 1
111. 0.17 mW/cm^2 for $m = 1$ only, no higher orders
113. $28.7°$
115. a. 42.3 nm; b. This wavelength is not in the visible spectrum. c. The number of slits in this diffraction grating is too large. Etching in integrated circuits can be done to a resolution of 50 nm, so slit separations of 400 nm are at the limit of what we can do today. This line spacing is too small to produce diffraction of light.
117. a. 549 km; b. This is an unreasonably large telescope. c. Unreasonable to assume diffraction limit for optical telescopes unless in space due to atmospheric effects.

CHALLENGE PROBLEMS

119. a. $\text{I} = 0.00500\,\text{I}_0,\ 0.00335\,\text{I}_0$; b. $\text{I} = 0.00500\,\text{I}_0,\ 0.00335\,\text{I}_0$
121. 12,800
123. 1.58×10^{-6} m

CHAPTER 5

CHECK YOUR UNDERSTANDING

5.1. Special relativity applies only to objects moving at constant velocity, whereas general relativity applies to objects that undergo acceleration.

5.2. $$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{(0.650c)^2}{c^2}}} = 1.32$$

5.3. a. $$\Delta t = \frac{\Delta\tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2.10 \times 10^{-8}\,\text{s}}{\sqrt{1 - \frac{(1.90 \times 10^8\ \text{m/s})^2}{(3.00 \times 10^8\ \text{m/s})^2}}} = 2.71 \times 10^{-8}\ \text{s}.$$

5.3. b. Only the relative speed of the two spacecraft matters because there is no absolute motion through space. The signal is emitted from a fixed location in the frame of reference of *A*, so the proper time interval of its emission is $\tau = 1.00\,\text{s}$. The duration of the signal measured from frame of reference B is then

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1.00\text{ s}}{\sqrt{1 - \frac{(4.00\times 10^7\text{ m/s})^2}{(3.00\times 10^8\text{ m/s})^2}}} = 1.01\text{ s}.$$

5.4. $L = L_0\sqrt{1 - \frac{v^2}{c^2}} = (2.50\text{ km})\sqrt{1 - \frac{(0.750c)^2}{c^2}} = 1.65\text{ km}$

5.5. Start with the definition of the proper time increment:

$$d\tau = \sqrt{-(ds)^2/c^2} = \sqrt{dt^2 - \left(dx^2 + dx^2 + dx^2\right)/c^2}.$$

where (dx, dy, dx, cdt) are measured in the inertial frame of an observer who does not necessarily see that particle at rest. This therefore becomes

$$\begin{aligned} d\tau &= \sqrt{-(ds)^2/c^2} = \sqrt{dt^2 - \left[(dx)^2 + (dy)^2 + (dz)^2\right]/c^2} \\ &= dt\sqrt{1 - \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]/c^2} \\ &= dt\sqrt{1 - v^2/c^2} \\ dt &= \gamma d\tau. \end{aligned}$$

5.6. Although displacements perpendicular to the relative motion are the same in both frames of reference, the time interval between events differ, and differences in dt and dt' lead to different velocities seen from the two frames.

5.7. We can substitute the data directly into the equation for relativistic Doppler frequency:

$$f_{\text{obs}} = f_s\sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} = (1.50\text{ GHz})\sqrt{\frac{1 - \frac{0.350c}{c}}{1 + \frac{0.350c}{c}}} = 1.04\text{ GHz}.$$

5.8. Substitute the data into the given equation:

$$p = \gamma mu = \frac{mu}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{(9.11\times 10^{-31}\text{ kg})(0.985)(3.00\times 10^8\text{ m/s})}{\sqrt{1 - \frac{(0.985c)^2}{c^2}}} = 1.56\times 10^{-21}\text{ kg-m/s}.$$

5.9.

$$\begin{aligned} K_{\text{rel}} &= (\gamma - 1)mc^2 = \left(\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1\right)mc^2 \\ &= \left(\frac{1}{\sqrt{1 - \frac{(0.992c)^2}{c^2}}} - 1\right)(9.11\times 10^{-31}\text{ kg})(3.00\times 10^8\text{ m/s})^2 = 5.67\times 10^{-13}\text{ J} \end{aligned}$$

CONCEPTUAL QUESTIONS

1. the second postulate, involving the speed of light; classical physics already included the idea that the laws of mechanics, at least, were the same in all inertial frames, but the velocity of a light pulse was different in different frames moving with respect to each other

3. yes, provided the plane is flying at constant velocity relative to the Earth; in that case, an object with no force acting on it within the plane has no change in velocity relative to the plane and no change in velocity relative to the Earth; both the plane and the ground are inertial frames for describing the motion of the object

5. The observer moving with the process sees its interval of proper time, which is the shortest seen by any observer.

7. The length of an object is greatest to an observer who is moving with the object, and therefore measures its proper length.

9. a. No, not within the astronaut's own frame of reference. b. He sees Earth clocks to be in their rest frame moving by him, and therefore sees them slowed. c. No, not within the astronaut's own frame of reference. d. Yes, he measures the distance between the two stars to be shorter. e. The two observers agree on their relative speed.

11. There is no measured change in wavelength or frequency in this case. The relativistic Doppler effect depends only on the relative velocity of the source and the observer, not any speed relative to a medium for the light waves.

13. It shows that the stars are getting more distant from Earth, that the universe is expanding, and doing so at an accelerating rate, with greater velocity for more distant stars.]

15. Yes. This can happen if the external force is balanced by other externally applied forces, so that the net external force is zero.

17. Because it loses thermal energy, which is the kinetic energy of the random motion of its constituent particles, its mass decreases by an extremely small amount, as described by energy-mass equivalence.
19. Yes, in principle there would be a similar effect on mass for any decrease in energy, but the change would be so small for the energy changes in a chemical reaction that it would be undetectable in practice.
21. Not according to special relativity. Nothing with mass can attain the speed of light.

PROBLEMS

23. a. 1.0328; b. 1.15
25. 5.96×10^{-8} s
27. 0.800c
29. 0.140c
31. 48.6 m
33. Using the values given in **Example 5.3**: a. 1.39 km; b. 0.433 km; c. 0.433 km
35. a. 10.0c; b. The resulting speed of the canister is greater than c, an impossibility. c. It is unreasonable to assume that the canister will move toward the earth at 1.20c.
37. The angle α approaches $45°$, and the t'- and x'-axes rotate toward the edge of the light cone.
39. 15 m/s east
41. 32 m/s
43. a. The second ball approaches with velocity $-v$ and comes to rest while the other ball continues with velocity $-v$; b. This conserves momentum.
45. a. $t_1{}' = 0;\ \ x_1{}' = 0;$ $t_2{}' = \tau;\ \ x_2{}' = 0$; b. $t_1{}' = 0;\ \ x_1{}' = 0;$ $t_2{}' = \dfrac{\tau}{\sqrt{1 - v^2/c^2}};\ \ x_2{}' = \dfrac{-v\tau}{\sqrt{1 - v^2/c^2}}$

47. 0.615c
49. 0.696c
51. (Proof)
53. 4.09×10^{-19} kg · m/s
55. a. $3.000000015 \times 10^{13}$ kg · m/s; b. 1.000000005
57. 2.988×10^{8} m/s
59. 0.512 MeV according to the number of significant figures stated. The exact value is closer to 0.511 MeV.
61. 2.3×10^{-30} kg; to two digits because the difference in rest mass energies is found to two digits
63. a. 1.11×10^{27} kg; b. 5.56×10^{-5}
65. a. 7.1×10^{-3} kg; b. $7.1 \times 10^{-3} = 7.1 \times 10^{-3}$; c. $\dfrac{\Delta m}{m}$ is greater for hydrogen
67. a. 208; b. 0.999988c; six digits used to show difference from c
69. a. 6.92×10^{5} J; b. 1.54
71. a. 0.914c; b. The rest mass energy of an electron is 0.511 MeV, so the kinetic energy is approximately 150% of the rest mass energy. The electron should be traveling close to the speed of light.

ADDITIONAL PROBLEMS

73. a. 0.866c; b. 0.995c
75. a. 4.303 y to four digits to show any effect; b. 0.1434 y; c. $1/\sqrt{\left(1 - v^2/c^2\right)} = 29.88$.
77. a. 4.00; b. $v = 0.867c$
79. a. A sends a radio pulse at each heartbeat to B, who knows their relative velocity and uses the time dilation formula to calculate the proper time interval between heartbeats from the observed signal. b. $(66 \text{ beats/min})\sqrt{1 - v^2/c^2} = 57.1 \text{ beats/min}$
81. a. first photon: $(0,\ 0,\ 0)$ at $t = t'$; second photon:

$$t' = \frac{-vx/c^2}{\sqrt{1 - v^2/c^2}} = \frac{-(c/2)(1.00 \text{ m})/c^2}{\sqrt{0.75}} = -\frac{0.577 \text{ m}}{c} = 1.93 \times 10^{-9} \text{ s}$$

$$x' = \frac{x}{\sqrt{1 - v^2/c^2}} = \frac{1.00 \text{ m}}{\sqrt{0.75}} = 1.15 \text{ m}$$

b. simultaneous in A, not simultaneous in B

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} = \frac{\left(4.5 \times 10^{-4}\,\text{s}\right) - (0.6c)\left(\frac{150 \times 10^3\ \text{m}}{c^2}\right)}{\sqrt{1 - (0.6)^2}}$$

$$= 1.88 \times 10^{-4}\ \text{s}$$

83.

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} = \frac{150 \times 10^3\ \text{m} - (0.60)\left(3.00 \times 10^8\ \text{m/s}\right)\left(4.5 \times 10^{-4}\ \text{s}\right)}{\sqrt{1 - (0.6)^2}}$$

$$= -1.01 \times 10^5\ \text{m} = -101\ \text{km}$$

$$y = y' = 15\ \text{km}$$

$$z = z' = 1\ \text{km}$$

85.

$$\Delta t = \frac{\Delta t' + v\Delta x'/c^2}{\sqrt{1 - v^2/c^2}}$$

$$0 = \frac{\Delta t' + v(500\ \text{m})/c^2}{\sqrt{1 - v^2/c^2}};$$

since $v \ll c,$ we can ignore the term v^2/c^2 and find

$$\Delta t' = -\frac{(50\ \text{m/s})(500\ \text{m})}{\left(3.00 \times 10^8\ \text{m/s}\right)^2} = -2.78 \times 10^{-13}\ \text{s}$$

The breakdown of Newtonian simultaneity is negligibly small, but not exactly zero, at realistic train speeds of 50 m/s.

87.

$$\Delta t' = \frac{\Delta t - v\Delta x/c^2}{\sqrt{1 - v^2/c^2}}$$

$$0 = \frac{(0.30\ \text{s}) - \dfrac{(v)\left(2.0 \times 10^9\ \text{m}\right)}{\left(3.00 \times 10^8\ \text{m/s}\right)^2}}{\sqrt{1 - v^2/c^2}}$$

$$v = \frac{(0.30\ \text{s})}{\left(2.0 \times 10^9\ \text{m}\right)}\left(3.00 \times 10^8\ \text{m/s}\right)^2$$

$$v = 1.35 \times 10^7\ \text{m/s}$$

89. Note that all answers to this problem are reported to five significant figures, to distinguish the results. a. 0.99947*c*; b. $1.2064 \times 10^{11}\ \text{y}$; c. $1.2058 \times 10^{11}\ \text{y}$

91. a. –0.400*c*; b. –0.909*c*

93. a. 1.65 km/s; b. Yes, if the speed of light were this small, speeds that we can achieve in everyday life would be larger than 1% of the speed of light and we could observe relativistic effects much more often.

95. 775 MHz

97. a. $1.12 \times 10^{-8}\ \text{m/s}$; b. The small speed tells us that the mass of a protein is substantially smaller than that of even a tiny amount of macroscopic matter.

99. a.

$$F = \frac{dp}{dt} = \frac{d}{dt}\left(\frac{mu}{\sqrt{1 - u^2/c^2}}\right)$$

$$= \frac{du}{dt}\left(\frac{m}{\sqrt{1 - u^2/c^2}}\right) - \frac{1}{2}\frac{mu^2}{\left(1 - u^2/c^2\right)^{3/2}}2\frac{du}{dt};$$

$$= \frac{m}{\left(1 - u^2/c^2\right)^{3/2}}\frac{du}{dt}$$

b.

$$\begin{aligned} F &= \frac{m}{\left(1 - u^2/c^2\right)^{3/2}} \frac{du}{dt} \\ &= \frac{1\text{ kg}}{\left(1 - \left(\frac{1}{2}\right)^2\right)^{3/2}} \left(1\text{ m/s}^2\right) \\ &= 1.53\text{ N} \end{aligned}$$

101. 90.0 MeV

103. a. $\gamma^2 - 1$; b. yes

105. 1.07×10^3

107. a. 6.56×10^{-8} kg; b. $m = (200\text{ L})\left(1\text{ m}^3/1000\text{ L}\right)\left(750\text{ kg/m}^3\right) = 150\text{ kg}$; therefore, $\frac{\Delta m}{m} = 4.37 \times 10^{-10}$

109. a. 0.314*c*; b. 0.99995*c* (Five digits used to show difference from *c*)

111. a. 1.00 kg; b. This much mass would be measurable, but probably not observable just by looking because it is 0.01% of the total mass.

113. a. 6.06×10^{11} kg/s; b. 4.67×10^{10} y; c. 4.27×10^9 kg; d. 0.32%

CHAPTER 6

CHECK YOUR UNDERSTANDING

6.1. Bunsen's burner

6.2. The wavelength of the radiation maximum decreases with increasing temperature.

6.3. $T_\alpha / T_\beta = 1/\sqrt{3} \cong 0.58,$ so the star β is hotter.

6.4. 3.3×10^{-19} J

6.5. No, because then $\Delta E/E \approx 10^{-21}$

6.6. -0.91 V; 1040 nm

6.7. $h = 6.40 \times 10^{-34}\,\text{J}\cdot\text{s} = 4.0 \times 10^{-15}\,\text{eV}\cdot\text{s}; -3.5\ \%$

6.8. $(\Delta\lambda)_{\text{min}} = 0\text{m}$ at a $0°$ angle; $71.0\text{ pm} + 0.5\lambda_c = 72.215\text{ pm}$

6.9. 121.5 nm and 91.1 nm; no, these spectral bands are in the ultraviolet

6.10. $v_2 = 1.1 \times 10^6\,\text{m/s} \cong 0.0036c$; $L_2 = 2\hbar$ $K_2 = 3.4\,\text{eV}$

6.11. 1.7 pm

6.12. $\lambda = 2\pi n a_0 = 2(3.324\text{ Å}) = 6.648\text{ Å}$

6.13. $\lambda = 1.417\text{ pm}$; $K = 261.56\text{ keV}$

6.14. $0.052°$

6.15. doubles it

CONCEPTUAL QUESTIONS

1. yellow

3. goes from red to violet through the rainbow of colors

5. would not differ

7. human eye does not see IR radiation

9. No

11. from the slope

13. Answers may vary

15. the particle character

17. Answers may vary

19. no; yes

21. no

23. right angle

25. no

27. They are at ground state.

29. Answers may vary

31. increase
33. for larger *n*
35. Yes, the excess of 13.6 eV will become kinetic energy of a free electron.
37. no
39. X-rays, best resolving power
41. proton
43. negligibly small de Broglie's wavelengths
45. to avoid collisions with air molecules
47. Answers may vary
49. Answers may vary
51. yes
53. yes

PROBLEMS

55. a. 0.81 eV; b. 2.1×10^{23}; c. 2 min 20 sec
57. a. 7245 K; b. 3.62 μm
59. about 3 K
61. 4.835×10^{18} Hz; 0.620 Å
63. 263 nm; no
65. 369 eV
67. 4.09 eV
69. 5.54 eV
71. a. 1.89 eV; b. 459 THz; c. 1.21 V
73. 264 nm; UV
75. 1.95×10^{6} m/s
77. 1.66×10^{-32} kg · m/s
79. 56.21 eV
81. 6.63×10^{-23} kg · m/s; 124 keV
83. 82.9 fm; 15 MeV
85. (Proof)
87. $\Delta\lambda_{30}/\Delta\lambda_{45} = 45.74\%$
89. 121.5 nm
91. a. 0.661 eV; b. –10.2 eV; c. 1.511 eV
93. 3038 THz
95. 97.33 nm
97. a. h/π; b. 3.4 eV; c. – 6.8 eV; d. – 3.4 eV
99. $n = 4$
101. 365 nm; UV
103. no
105. 7
107. 145.5 pm
109. 20 fm; 9 fm
111. a. 2.103 eV; b. 0.846 nm
113. 80.9 pm
115. 2.21×10^{-20} m/s
117. 9.929×10^{32}
119. $\gamma = 1060$; 0.00124 fm
121. 24.11 V
123. a. $P = 2I/c = 8.67 \times 10^{-6}\,\text{N/m}^2$; b. $a = PA/m = 8.67 \times 10^{-4}\,\text{m/s}^2$; c. 74.91 m/s
125. $x = 4.965$

ADDITIONAL PROBLEMS

127. $7.124 \times 10^{16}\,\text{W/m}^3$
129. 1.034 eV

131. 5.93×10^{18}
133. 387.8 nm
135. a. 4.02×10^{15}; b. 0.533 mW
137. a. 4.02×10^{15}; b. 0.533 mW; c. 0.644 mA; d. 2.57 ns
139. a. 0.132 pm; b. 9.39 MeV; c. 0.047 MeV
141. a. 2 kJ; b. 1.33×10^{-5} kg · m/s; c. 1.33×10^{-5} N; d. yes
143. a. 0.003 nm; b. $105.56°$
145. $n = 3$
147. a. $a_0/2$; b. $-54.4\ \text{eV}/n^2$; c. $a_0/3$, $-122.4\ \text{eV}/n^2$
149. a. 36; b. 18.2 nm; c. UV
151. 396 nm; 5.23 neV
153. 7.3 keV
155. 728 m/s; $1.5\mu\text{V}$
157. $\lambda = hc/\sqrt{K(2E_0 + K)} = 3.705\ \text{nm}, K = 100\ \text{keV}$
159. $\Delta\lambda_c^{(\text{electron})}/\Delta\lambda_c^{(\text{proton})} = m_p/m_e = 1836$
161. (Proof)
163. $5.1 \times 10^{17}\ \text{Hz}$

CHAPTER 7

CHECK YOUR UNDERSTANDING

7.1. $(3 + 4i)(3 - 4i) = 9 - 16i^2 = 25$
7.2. $A = \sqrt{2/L}$
7.3. $(1/2 - 1/\pi)/2 = 9\%$
7.4. $4.1 \times 10^{-8}\ \text{eV}$; $1.1 \times 10^{-5}\ \text{nm}$
7.5. $0.5m\omega^2 x^2 \psi(x)^* \psi(x)$
7.6. None. The first function has a discontinuity; the second function is double-valued; and the third function diverges so is not normalizable.
7.7. a. 9.1%; b. 25%
7.8. a. 295 N/m; b. 0.277 eV
7.9. $\langle x \rangle = 0$
7.10. $L_{\text{proton}}/L_{\text{electron}} = \sqrt{m_e/m_p} = 2.3\%$

CONCEPTUAL QUESTIONS

1. $1/\sqrt{L}$, where $L = \text{length}$; $1/L$, where $L = \text{length}$
3. The wave function does not correspond directly to any measured quantity. It is a tool for predicting the values of physical quantities.
5. The average value of the physical quantity for a large number of particles with the same wave function.
7. Yes, if its position is completely unknown. Yes, if its momentum is completely unknown.
9. No. According to the uncertainty principle, if the uncertainty on the particle's position is small, the uncertainty on its momentum is large. Similarly, if the uncertainty on the particle's position is large, the uncertainty on its momentum is small.
11. No, it means that predictions about the particle (expressed in terms of probabilities) are time-independent.
13. No, because the probability of the particle existing in a narrow (infinitesimally small) interval at the discontinuity is undefined.
15. No. For an infinite square well, the spacing between energy levels increases with the quantum number *n*. The *smallest* energy measured corresponds to the transition from *n* = 2 to 1, which is three times the ground state energy. The largest *energy* measured corresponds to a transition from $n = \infty$ to 1, which is infinity. (Note: Even particles with extremely large energies remain bound to an infinite square well—they can never "escape")
17. No. This energy corresponds to $n = 0.25$, but *n* must be an integer.
19. Because the smallest allowed value of the quantum number *n* for a simple harmonic oscillator is 0. No, because quantum mechanics and classical mechanics agree only in the limit of large n.

21. Yes, within the constraints of the uncertainty principle. If the oscillating particle is localized, the momentum and therefore energy of the oscillator are distributed.
23. doubling the barrier width
25. No, the restoring force on the particle at the walls of an infinite square well is infinity.

PROBLEMS

27. $\left|\psi(x)\right|^2 \sin^2 \omega t$

29. (a) and (e), can be normalized
31. a. $A = \sqrt{2\alpha/\pi}$; b. $\text{probability} = 29.3\%$; c. $\langle x \rangle = 0$; d. $\langle p \rangle = 0$; e. $\langle K \rangle = \alpha^2 \hbar^2/2m$

33. a. $\Delta p \geq 2.11 \times 10^{-34}\,\text{N}\cdot\text{s}$; b. $\Delta v \geq 6.31 \times 10^{-8}\,\text{m}$; c. $\Delta v/\sqrt{k_B T/m_\alpha} = 5.94 \times 10^{-11}$

35. $\Delta\tau \geq 1.6 \times 10^{-25}\,\text{s}$

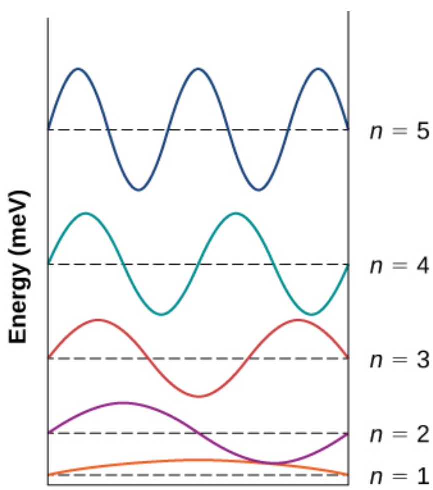
37. a. $\Delta f \geq 1.59\,\text{MHz}$; b. $\Delta\omega/\omega_0 = 3.135 \times 10^{-9}$

39. Carrying out the derivatives yields $k^2 = \frac{\omega^2}{c^2}$.

41. Carrying out the derivatives (as above) for the sine function gives a cosine on the right side the equation, so the equality fails. The same occurs for the cosine solution.
43. $E = \hbar^2 k^2/2m$

45. $\hbar^2 k^2$; The particle has definite momentum and therefore definite momentum squared.

47. 9.4 eV, 64%
49. 0.38 nm
51. 1.82 MeV
53. 24.7 nm
55. 6.03 Å
57. a.



;

b. $\lambda_{5 \to 3} = 12.9\,\text{nm}$, $\lambda_{3 \to 1} = 25.8\,\text{nm}$, $\lambda_{4 \to 3} = 29.4\,\text{nm}$
59. proof
61. $6.662 \times 10^{14}\,\text{Hz}$

63. $n \approx 2.037 \times 10^{30}$

65. $\langle x \rangle = 0.5m\omega^2 \langle x^2 \rangle = \hbar\omega/4$; $\langle K \rangle = \langle E \rangle - \langle U \rangle = \hbar\omega/4$

67. proof
69. A complex function of the form, $Ae^{i\phi}$, satisfies Schrӧdinger's time-independent equation. The operators for kinetic and total energy are linear, so any linear combination of such wave functions is also a valid solution to Schrӧdinger's equation. Therefore, we conclude that **Equation 7.113** satisfies **Equation 7.106**, and **Equation 7.114** satisfies **Equation 7.108**.
71. a. 4.21%; b. 0.84%; c. 0.06%

73. a. 0.13%; b. close to 0%
75. 0.38 nm

ADDITIONAL PROBLEMS

77. proof
79. a. 4.0 %; b. 1.4 %; c. 4.0%; d. 1.4%
81. a. $t = mL^2/h = 2.15 \times 10^{26}$ years ; b. $E_1 = 1.46 \times 10^{-66}$ J,$K = 0.4$ J
83. proof
85. 1.2 N/m
87. 0

CHALLENGE PROBLEMS

89. $19.2\ \mu$m; $11.5\ \mu$m
91. 3.92%
93. proof

CHAPTER 8

CHECK YOUR UNDERSTANDING

8.1. No. The quantum number $m = -l,\ -l+1, \ldots,\ 0, \ldots,\ l-1,\ l.$ Thus, the magnitude of L_z is always less than L because $< \sqrt{l(l+1)}$
8.2. $s = 3/2 <$
8.3. frequency quadruples

CONCEPTUAL QUESTIONS

1. *n* (principal quantum number) $\rightarrow$ total energy
l (orbital angular quantum number) $\rightarrow$ total absolute magnitude of the orbital angular momentum
m (orbital angular projection quantum number) $\rightarrow$ *z*-component of the orbital angular momentum
3. The Bohr model describes the electron as a particle that moves around the proton in well-defined orbits. Schrödinger's model describes the electron as a wave, and knowledge about the position of the electron is restricted to probability statements. The total energy of the electron in the ground state (and all excited states) is the same for both models. However, the orbital angular momentum of the ground state is different for these models. In Bohr's model, $L(\text{ground state}) = 1$, and in Schrödinger's model, $L(\text{ground state}) = 0$.
5. a, c, d; The total energy is changed (Zeeman splitting). The work done on the hydrogen atom rotates the atom, so the *z*-component of angular momentum and polar angle are affected. However, the angular momentum is not affected.
7. Even in the ground state ($l = 0$), a hydrogen atom has magnetic properties due the intrinsic (internal) electron spin. The magnetic moment of an electron is proportional to its spin.
9. For all electrons, $s = ½$ and $m_s = \pm ½.$ As we will see, not all particles have the same spin quantum number. For example, a photon as a spin 1 ($s = 1$), and a Higgs boson has spin 0 ($s = 0$).
11. An electron has a magnetic moment associated with its intrinsic (internal) spin. Spin-orbit coupling occurs when this interacts with the magnetic field produced by the orbital angular momentum of the electron.
13. Elements that belong in the same column in the periodic table of elements have the same fillings of their outer shells, and therefore the same number of valence electrons. For example:
Li: $1s^2 2s^1$ (one valence electron in the $n = 2$ shell)
Na: $1s^2 2s 2p^6 3s^1$ (one valence electron in the $n = 2$ shell)
Both, Li and Na belong to first column.
15. Atomic and molecular spectra are said to be "discrete," because only certain spectral lines are observed. In contrast, spectra from a white light source (consisting of many photon frequencies) are continuous because a continuous "rainbow" of colors is observed.
17. UV light consists of relatively high frequency (short wavelength) photons. So the energy of the absorbed photon and the energy transition (ΔE) in the atom is relatively large. In comparison, visible light consists of relatively lower-frequency photons. Therefore, the energy transition in the atom and the energy of the emitted photon is relatively small.
19. For macroscopic systems, the quantum numbers are very large, so the energy difference (ΔE) between adjacent energy levels (orbits) is very small. The energy released in transitions between these closely space energy levels is much too small to be detected.

21. Laser light relies on the process of stimulated emission. In this process, electrons must be prepared in an excited (upper) metastable state such that the passage of light through the system produces de-excitations and, therefore, additional light.
23. A Blu-Ray player uses blue laser light to probe the bumps and pits of the disc and a CD player uses red laser light. The relatively short-wavelength blue light is necessary to probe the smaller pits and bumps on a Blu-ray disc; smaller pits and bumps correspond to higher storage densities.

PROBLEMS

25. $(r, \theta, \phi) = (\sqrt{6}, 66°, 27°)$.

27. $\pm 3, \pm 2, \pm 1, 0$ are possible

29. $\pm 3, \pm 2, \pm 1, 0$ are possible

31. $F = -k\frac{Qq}{r^2}$

33. (1, 1, 1)
35. For the orbital angular momentum quantum number, *l*, the allowed values of:
$m = -l, -l + 1, \ldots 0, \ldots l - 1, l$.

With the exception of $m = 0$, the total number is just 2*l* because the number of states on either side of $m = 0$ is just *l*. Including $m = 0$, the total number of orbital angular momentum states for the orbital angular momentum quantum number, *l*, is: $2l + 1$. Later, when we consider electron spin, the total number of angular momentum states will be found to twice this value because each orbital angular momentum states is associated with two states of electron spin: spin up and spin down).

37. The probability that the 1*s* electron of a hydrogen atom is found outside of the Bohr radius is $\int_{a_0}^{\infty} P(r)dr \approx 0.68$

39. For $n = 2$, $l = 0$ (1 state), and $l = 1$ (3 states). The total is 4.

41. The 3*p* state corresponds to $n = 3$, $l = 2$. Therefore, $\mu = \mu_B \sqrt{6}$

43. The ratio of their masses is 1/207, so the ratio of their magnetic moments is 207. The electron's magnetic moment is more than 200 times larger than the muon.
45. a. The 3*d* state corresponds to $n = 3$, $l = 2$. So,

$I = 4.43 \times 10^{-7}$ A.

b. The maximum torque occurs when the magnetic moment and external magnetic field vectors are at right angles ($\sin\theta = 1$) . In this case:

$|\vec{\tau}| = \mu B.$

$\tau = 5.70 \times 10^{-26}$ N · m.

47. A 3*p* electron is in the state $n = 3$ and $l = 1$. The minimum torque magnitude occurs when the magnetic moment and external magnetic field vectors are most parallel (antiparallel). This occurs when $m = \pm 1$ .The torque magnitude is given by

$|\vec{\tau}| = \mu B \sin\theta,$

Where

$\mu = (1.31 \times 10^{-24}$ J/T).

For $m = \pm 1,$ we have:

$|\vec{\tau}| = 2.32 \times 10^{21}$ N · m.

49. An infinitesimal work *dW* done by a magnetic torque τ to rotate the magnetic moment through an angle $-d\theta$:
$dW = \tau(-d\theta)$,

where $\tau = |\vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}}|$. Work done is interpreted as a drop in potential energy *U*, so

$dW = -dU.$

The total energy change is determined by summing over infinitesimal changes in the potential energy:
$U = -\mu B \cos\theta$

$U = -\vec{\boldsymbol{\mu}} \cdot \vec{\mathbf{B}}$.

51. Spin up (relative to positive *z*-axis):
$\theta = 55°.$

Spin down (relative to positive *z*-axis):

$$\theta = \cos^{-1}\left(\frac{S_z}{S}\right) = \cos^{-1}\left(\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{3}}\right) = 125°.$$

53. The spin projection quantum number is $m_s = \pm ½$, so the *z*-component of the magnetic moment is $\mu_z = \pm\mu_B$.

The potential energy associated with the interaction between the electron and the external magnetic field is $U = \mp\mu_B B$.

The energy difference between these states is $\Delta E = 2\mu_B B$, so the wavelength of light produced is

$$\lambda = 8.38 \times 10^{-5} \text{ m} \approx 84\ \mu\text{m}$$

55. It is increased by a factor of 2.

57. a. 32; b.

ℓ		$2(2\ell+1)$		
0	*s*	$2(0+1)$	=	2
1	*p*	$2(2+1)$	=	6
2	*d*	$2(4+1)$	=	10
3	*f*	$2(6+1)$	=	14
				32

59. a. and e. are allowed; the others are not allowed.

b. $l = 3$ not allowed for $n = 1,\ l \le (n-1)$.

c. Cannot have three electrons in *s* subshell because $3 > 2(2l+1) = 2$.

d. Cannot have seven electrons in *p* subshell (max of 6) $2(2l+1) = 2(2+1) = 6$.

61. $[\text{Ar}]\, 4s^2\, 3d^6$

63. a. The minimum value of ℓ is $l = 2$ to have nine electrons in it.

b. $3d^9$.

65. $[\text{He}]\, 2s^2\, 2p^2$

67. For He^+, one electron "orbits" a nucleus with two protons and two neutrons ($Z = 2$). Ionization energy refers to the energy required to remove the electron from the atom. The energy needed to remove the electron in the ground state of He^+ ion to infinity is negative the value of the ground state energy, written:

$$E = -54.4 \text{ eV}.$$

Thus, the energy to ionize the electron is $+54.4$ eV.

Similarly, the energy needed to remove an electron in the first excited state of Li^{2+} ion to infinity is negative the value of the first excited state energy, written:

$$E = -30.6 \text{ eV}.$$

The energy to ionize the electron is 30.6 eV.

69. The wavelength of the laser is given by:

$$\lambda = \frac{hc}{-\Delta E},$$

where E_γ is the energy of the photon and ΔE is the magnitude of the energy difference. Solving for the latter, we get:

$$\Delta E = -2.795 \text{ eV}.$$

The negative sign indicates that the electron lost energy in the transition.

71. $\Delta E_{L \to K} \approx (Z-1)^2 (10.2 \text{ eV}) = 3.68 \times 10^3 \text{ eV}.$

73. According to the conservation of the energy, the potential energy of the electron is converted completely into kinetic energy. The initial kinetic energy of the electron is zero (the electron begins at rest). So, the kinetic energy of the electron just before it strikes the target is:

$$K = e\Delta V.$$

If *all* of this energy is converted into braking radiation, the frequency of the emitted radiation is a maximum, therefore:

$$f_{\max} = \frac{e\Delta V}{h}.$$

When the emitted frequency is a maximum, then the emitted wavelength is a minimum, so:
$\lambda_{\text{min}} = 0.1293 \text{ nm}.$

75. A muon is 200 times heavier than an electron, but the minimum wavelength does not depend on mass, so the result is unchanged.
77. 4.13×10^{-11} m
79. 72.5 keV
81. The atomic numbers for Cu and Au are $Z = 29$ and 79, respectively. The X-ray photon frequency for gold is greater than copper by a factor:

$$\left(\frac{f_{\text{Au}}}{f_{\text{Cu}}}\right)^2 = \left(\frac{79-1}{29-1}\right)^2 \approx 8.$$

Therefore, the X-ray wavelength of Au is about eight times shorter than for copper.
83. a. If flesh has the same density as water, then we used 1.34×10^{23} photons. b. 2.52 MW

ADDITIONAL PROBLEMS

85. The smallest angle corresponds to $l = n - 1$ and $m = l = n - 1$. Therefore $\theta = \cos^{-1}\left(\sqrt{\frac{n-1}{n}}\right)$.

87. a. According to **Equation 8.1**, when $r = 0$, $U(r) = -\infty$, and when $r = +\infty$, $U(r) = 0.$ b. The former result suggests that the electron can have an infinite negative potential energy. The quantum model of the hydrogen atom avoids this possibility because the probability density at $r = 0$ is zero.
89. A formal solution using sums is somewhat complicated. However, the answer easily found by studying the mathematical pattern between the principal quantum number and the total number of orbital angular momentum states.
For $n = 1$, the total number of orbital angular momentum states is 1; for $n = 2$, the total number is 4; and, when $n = 3$, the total number is 9, and so on. The pattern suggests the total number of orbital angular momentum states for the *n*th shell is n^2.
(Later, when we consider electron spin, the total number of angular momentum states will be found to be $2n^2$, because each orbital angular momentum states is associated with two states of electron spin; spin up and spin down).
91. 50
93. The maximum number of orbital angular momentum electron states in the *n*th shell of an atom is n^2. Each of these states can be filled by a spin up and spin down electron. So, the maximum number of electron states in the *n*th shell is $2n^2$.
95. a., c., and e. are allowed; the others are not allowed. b. $l > n$ is not allowed.
d. $7 > 2(2l + 1)$
97. $f = 1.8 \times 10^9$ Hz
99. The atomic numbers for Cu and Ag are $Z = 29$ and 47, respectively. The X-ray photon frequency for silver is greater than copper by the following factor:

$$\left(\frac{f_{\text{Ag}}}{f_{\text{Cu}}}\right)^2 = 2.7.$$

Therefore, the X-ray wavelength of Ag is about three times shorter than for copper.
101. a. 3.24; b. n_i is not an integer. c. The wavelength must not be correct. Because $n_i > 2,$ the assumption that the line was from the Balmer series is possible, but the wavelength of the light did not produce an integer value for n_i. If the wavelength is correct, then the assumption that the gas is hydrogen is not correct; it might be sodium instead.

CHAPTER 9

CHECK YOUR UNDERSTANDING

9.1. It corresponds to a repulsive force between core electrons in the ions.
9.2. the moment of inertia
9.3. more difficult
9.4. It decreases.
9.5. The forward bias current is much larger. To a good approximation, diodes permit current flow in only one direction.
9.6. a low temperature and low magnetic field

CONCEPTUAL QUESTIONS

59. a. 33.9 MeV; b. By conservation of momentum, $|p_\mu| = |p_\nu| = p$. By conservation of energy, $E_\nu = 29.8\ \text{MeV},\ E_\mu = 4.1\ \text{MeV}$

61. $(0.99)(299792\ \text{km/s}) = \left(\left(70\frac{\text{km}}{\text{s}}\right)/\ \text{Mpc}\right)(d),\ \ d = 4240\ \text{Mpc}$

63. 1.0×10^4 km/s away from us.

65. 2.26×10^8 y

67. a. $1.5 \times 10^{10}\ \text{y} = 15\ \text{billion years}$; b. Greater, since if it was moving slower in the past it would take less more to travel the distance.

69. $v = \sqrt{\frac{GM}{r}}$

ADDITIONAL PROBLEMS

71. a. $\bar{n}$; b. K^+; c. K^+; d. π^-; e. $\bar{\nu}_\tau$; f. e^+

73. 14.002 TeV ≈ 14.0 TeV

75. 964 rev/s

77. a. $H_0 = \frac{30\ \text{km/s}}{1\ \text{Mly}} = 30\text{km/s}\cdot\text{Mly}$; b. $H_0 = \frac{15\ \text{km/s}}{1\ \text{Mly}} = 15\text{km/s}\cdot\text{Mly}$

CHALLENGE PROBLEMS

79. a. 5×10^{10}; b. divide the number of particles by the area they hit: $5 \times 10^4\ \text{particles/m}^2$

81. a. 2.01; b. 2.50×10^{-8} s; c. 6.50 m

83.

$$\frac{mv^2}{r} = \frac{GMm}{r^2} \Rightarrow$$

$$v = \left(\frac{GM}{r}\right)^{1/2} = \left[\frac{\left(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\left(3 \times 10^{41}\ \text{kg}\right)}{(30{,}000\ \text{ly})\left(9.46 \times 10^{15}\ \text{m/ly}\right)}\right] = 2.7 \times 10^5\ \text{m/s}$$

85. a. 938.27 MeV; b. 1.84×10^3

87. a. $3.29 \times 10^{18}\ \text{GeV} \approx 3 \times 10^{18}\ \text{GeV}$; b. 0.3; Unification of the three forces breaks down shortly after the separation of gravity from the unification force (near the Planck time interval). The uncertainty in time then becomes greater. Hence the energy available becomes less than the needed unification energy.

INDEX

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